Restitution in Point Collisions

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Abstract

In the classical formulation of the dynamics of colliding objects, rebound in a direction normal to the contacting surfaces is modelled using the well known coefficient of restitution. This coefficient is usually defined (kinematically) as the ratio of the final relative normal velocity to the initial relative normal velocity of a point in the contact region. Another definition is the ratio of the normal impulse during rebound to the normal impulse during approach, a kinetic definition. These two definitions generally are the same but differ in some important cases. Both definitions however serve the same purpose in an impact analysis, as an artifice for kinetic energy loss. As such, each provides an alternative approach to the formulation of an impact analysis, each with its own advantages and disadvantages. These are discussed in the context of an impact of a rigid body with a massive surface.

Evidence exists that restitution can occur in tangential and torsional modes during collisions of rigid bodies. The classical model is extended in this work to include these phenomena by appropriate definition of new coefficients. This results in bilinear models for velocity-change dynamics. These models are assessed using experimental data. An analytical relationship of these coefficients to kinetic energy loss is provided.

1 Introduction

Analysis of the mechanics of impact can be a very complicated subject. Applications can require linear and nonlinear combinations of friction, thermal effects, dynamic plasticity, wave propagation, etc. Even low speed collisions can be complicated, although finite element analysis techniques now provide expanded solution capabilities. In some problems, the main interest lies not in the stresses and displacements in the contact region but rather the dynamics (velocity changes and energy loss) of the colliding objects. For many of these problems the classical approach using concepts of impulse and momentum is not only sufficient, but because of its simplicity, greatly advantageous. For example, when a designer is interested in the effect upon a control strategy of a collision of a robot's end effector, a simple model of the collision is desirable. A simple model is usually sufficient to model chaotic dynamics of vibratory impact. If an automotive engineer wishes to simulate highway speed collisions for arbitrary vehicle orientations, he or she will quickly find the finite element approach impractical.

In all cases where a simple model of impact is needed, the classical approach is attractive. Unfortunately, it has been deprecated and neglected for many years as being oversimplified and impractical. The classical approach uses coefficients such as the coefficients of restitution and friction in place of more exact dynamic analyses of local material behavior. What is often overlooked however is that analysis of the local material behavior or experiments can be used to determine a practical range of coefficient values. For example, Koller and Busenhart [1986] show by experiment and analysis how the coefficient of restitution varies for the elastic impact of spheres on thin shallow spherical shells. Another example is Shivakumar, *et al* [1985] for impacts of spheres on composite plates. Once the behavior for normal velocity changes is known, the classical approach provides an algebraic model for simulating velocity changes including other effects such as friction and surface geometry. The classical approach to impact dynamics using coefficients, and the more exact studies of local dynamic material behavior, should be viewed not as alternatives but as complementary techniques.

The classical concept of restitution, its definition and its relationship to energy loss is reviewed. Following this, the experimental and analytical work of others concerning shearing effects during low speed impact are used as a basis for extending classical theory. Maw, Barber and Fawcett [1977, 1981] report on their experiments and analytical work which demonstrate tangential elastic effects during impacts of metal discs. Horak [1948] likewise describes torsional elastic effects. It is an intent in this paper to show how the concept of restitution and the use of a coefficient applies to motion other than normal during impacts. The classical model is extended to permit velocity changes of rigid bodies[1] to be calculated in the presence of non-normal restitution. The next section contains a review of the planar problem of impact of a rigid body with a massive barrier.

2 Planar Impact, Classical Approach

A summary of some pertinent results concerning planar, rigid body barrier impacts from Brach [1988] is presented in this section. The effects of non-normal restitution are not introduced until a later section. Figure 1 shows a free body diagram of an arbitrarily shaped body (lamina) of mass m acted upon by impulses P_t, P_n and M. The quantities P_t and P_n are the tangential and normal impulse components, respectively, of the resultant impulse P created by the collision. The quantity M is the impulse of the moment about point C of the distributed force whose resultant is the impulse P. The idealized case is treated where contact is primarily at a known point and so the moment impulse M can be taken as zero. Three unknowns exist for this problem. They are the final mass center velocity components,[2] V_n and V_t, and the final angular velocity Ω. Three fundamental equations can be written. These are

$$V_n + d_c \Omega = -e(v_n + d_c \omega) \qquad (1)$$

$$\mu V_n - V_t = \mu v_n - v_t \qquad (2)$$

and

$$m d_c V_n - m d_d V_t - I\Omega = m d_c v_n - m d_d v_t - I\omega \qquad (3)$$

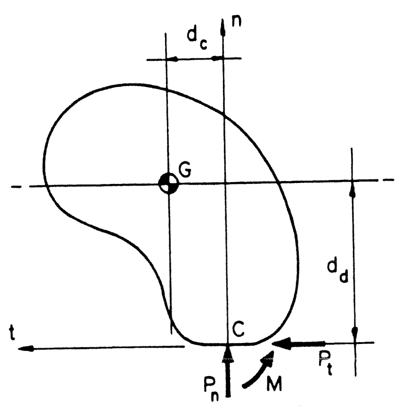

Figure 1. Free body diagram of an arbitrarily shaped rigid body (lamina) in planar impact.

Equation 1 follows from the definition of the coefficient of restitution; for example see Pestel and Thomson [1968]. Typically the range of e is restricted to $0 \le e \le 1$. The second equation is a consequence of the definition of μ as the impulse ratio,

$$\mu = P_t / P_n \qquad (4)$$

[1]The term rigid body is used here to indicate the presence of significant rotational inertia, not inflexibility.
[2]Note that the capital or upper case symbols represent final velocity components and small or lower case symbols represent initial velocity components.

which gives Equation (2), after the impulses P_t and P_n are replaced by their corresponding momentum changes. Note that μ is considered to be a constant which depends on the physical process which generates the tangential force system. Equation (4) is general enough to represent a tangential impulse arising from

1. Coulomb or dry friction,
2. indentation,
3. restitution, or
4. other forms of tangential force generation.

Finally, Equation (3) is an expression of conservation of angular momentum about point C.

Equations (1), (2) and (3) can be solved for the unknown final velocities. This solution yields:

$$V_n = v_n - \frac{1+e}{D}(v_n + d_c\omega) \tag{5}$$

$$V_t = v_t - \mu \frac{1+e}{D}(v_n + d_c\omega), \qquad |\mu| < |\mu_c| \tag{6}$$

$$\Omega = \omega - \frac{d_c - \mu d_d}{k^2}\frac{1+e}{D}(v_n + d_c\omega), \qquad |\mu| < |\mu_c| \tag{7}$$

where k is a centroidal radius of gyration defined by $k^2 = I/m$ and $D = 1 + d_c(d_c - \mu d_d)/k^2$. The role of μ and the selection of its appropriate value for the solution equations is not always obvious. If the tangential force is due only to Coulomb friction with coefficient f, then f is used in the solution equations in place of μ. However, for given initial conditions, it may happen that a tangential impulse P_t smaller than $f P_n$ causes sliding to end prior to separation. Then the value of μ corresponding to this smaller impulse must then be used. This type of behavior points out the need for a limiting, or critical, value μ_c which is discussed in the following paragraphs. Other possibilities may arise and are discussed by Brach [1989].

Energy lost during a collision is an important quantity. From the above equations, the energy loss T_L can be expressed as

$$T_L = \frac{m}{2}(v_n + d_c\omega)^2 \frac{1+e}{D}\left\{ 2 + 2\mu r - \frac{1+e}{D}\left[1 + \mu^2 + \frac{(d_c - \mu d_d)^2}{k^2}\right]\right\} \tag{8}$$

where $r = (v_t - d_d\omega)/(v_n + d_c\omega)$. For convenience this is often normalized by the maximum total energy loss, T_M, of the rigid body such that $T_L^* = T_L / T_M$; see Brach [1989] and the Appendix.

In the general case of planar rigid body impact with Coulomb friction, Keller [1986], Brach [1989] and others recognize that tangential velocity reversals at the contact surface can occur due to kinematical effects. Keller [1986] integrates the equations of motion to examine the conditions under which the velocity reversals occur. Ivanov [1986] also discusses the effects of friction during collisions. Brach [1989] examines the energy loss and whether or not the frictional force does positive work on the system. His view is that friction can only reduce the tangential velocity change which occurs in the frictionless case. That is, if a tangential velocity change occurs when a rigid body impacts a frictionless surface and the same impact with friction is considered, then a value of μ cannot be used which will cause the latter final tangential velocity to be greater than the frictionless case. Furthermore, a value of μ realistically can never be used in the solution equations Equations (5), (6) and (7), which can cause the kinetic energy of the rigid body to increase as the result of an impact. These considerations give rise to the critical impulse ratio.

2.1 Classical Coefficient Of Restitution

The notion of *coefficient of restitution* is attributed by Goldsmith [1960] directly to Isaac Newton. The kinematic definition of e, Equation (1), is an insightful expedient. The most obvious alternative for Equation (1) would be an expression for energy loss such as

$$T_L = T_i - T_f \tag{1-a}$$

where T_i and T_f are the initial and final kinetic energies, respectfully. Because energy is quadratic in the velocities, the impact problem formulated with Equation (1-a) in place of Equation (1) remains algebraic but becomes nonlinear. Consequently, the use of e as an energy loss parameter has the advantage of linearity over T_L and is completely equivalent as discussed in the Appendix.

The coefficient of restitution can be defined in a fashion different from Equation (1). The most common alternative is a kinetic definition, denoted here by R, where

$$R = P_n^R / P_n^A \tag{9}$$

as described by Christie [1964]. Here, P_n^R is the rebound or expansion portion of the normal impulse P_n and P_n^A is the approach or compression portion. The terms compression and expansion refer to the deformation associated with the normal contact forces. Thus

$$P_n = P_n^R + P_n^A \tag{10}$$

Christie [1964] shows that for point mass collisions, the kinematic and kinetic definitions give identical results, that is, $R \equiv e$. Unfortunately, for rigid body collisions, the kinematic and kinetic coefficients of restitution are not always identical. To show this, and to use the kinetic definition, requires breaking the contact duration into two parts, τ_1 to $\bar{\tau}$ and $\bar{\tau}$ to τ_2, where τ_1 is the time of initiation of contact, τ_2 is the end of contact (time of separation) and $\bar{\tau}$ is the time when compression ends and rebound begins. This requires the introduction of additional unknowns (velocities and impulses) and equations.

For a collision of a single rigid body as Fig. 1, it is shown in the Appendix that

$$R = e \frac{k^2 + d_c (d_c - \mu_A d_d)}{k^2 + d_c (d_c - \mu_R d_d)} \tag{11}$$

where μ_A and μ_R are partial impulse ratios separated into approach and rebound phases,

$$\mu_A = P_t^A / P_n^A \tag{12}$$

and

$$\mu_R = P_t^R / P_n^R \tag{13}$$

Since relative tangential motion can end at any time during the contact interval τ_1 to τ_2, it is apparent that μ_A and μ_R are not generally equal. From Equation (11), the kinematic and kinetic coefficients are the same when

- $\mu_A = \mu_R$
- $d_c - \mu_R d_d = d_c - \mu_A d_d$
- $d_c = 0$
- $d_d = 0$

A subset of the condition $\mu_A = \mu_R$ is a frictionless collision such that both are zero. Another important special case is when sliding continues without reversal throughout the contact duration in the presence of Coulomb friction. In this case $\mu_A = \mu_R = f$, where f is the friction coefficient.

Some of the implications of the above can be summarized as

1. The coefficient of restitution is an artifice for the energy loss parameter, T_L;
2. The use of the kinetic coefficient of restitution requires the introduction of additional unknown velocities and impulses (in effect, doubling the number of unknowns);
3. An equivalence exists between the two coefficients R and e such as given by Equation (11)

The practical significance of this for impact problems is that the kinematic coefficient of restitution, e, should generally be used. Use of the kinetic coefficient, R, may provide advantages in some instances and it can be viewed as a viable alternative.

2.2 Oblique, Central Impacts

A special case is considered in this section, where $d_c = 0$ and $d_d = \rho$. This is where the mass center lies directly above the contact point. This is done not only to simplify the discussion but because under these conditions (and when the tangential force is due to Coulomb friction alone) a velocity reversal at the contact point cannot occur. In this case the critical impulse ratio is

$$\mu_c = \frac{1}{1+\lambda}\ \frac{r}{1+e} \tag{14}$$

where $\lambda = k^2/\rho^2$ and μ_c is the impulse ratio which reduces the tangential velocity at the contact point to zero. The quantity μ_c bounds the impulse ratio; consequently, a critical tangential impulse exists which is $\mu_c P_n$. Brach [1984] shows that under the conditions of this Section, μ_c also corresponds to the value of μ which maximizes $T_L(\mu)$ and with $e = 0$ provides T_M. Specific bounds on the energy loss are pursued in more detail in the Appendix.

An example is now presented to provide a comparison with the response including tangential restitution. A circular disc with radius ρ has a radius of gyration of $k = \rho/\sqrt{2}$. For arbitrary initial velocities, v_n, v_t and ω, the final rebound velocities are found from Equations (5), (6) and (7) to be

$$V_n = -ev_n \tag{15}$$

$$V_t = v_t - \mu(1+e)\,v_n\,, \qquad |\mu| < |\mu_c| \tag{16}$$

$$\Omega = \omega + 2\mu(1+e)\,v_n/\rho\,, \qquad |\mu| < |\mu_c| \tag{17}$$

where Equation (14) provides the corresponding critical value of μ. Let V_{ct} and v_{ct} be the final and initial velocities at the contact point C. From Equations (16) and (17),

$$V_{ct} = v_{ct} - \mu(1+e)\,v_n(1+\lambda) \tag{18}$$

or, using Equation (14),

$$V_{ct} = v_{ct}\left(1 - \frac{\mu}{\mu_c}\right), \qquad |\mu| \le |\mu_c| \tag{19}$$

The normalized final velocity V_{ct}/v_n is plotted in Fig. 2.

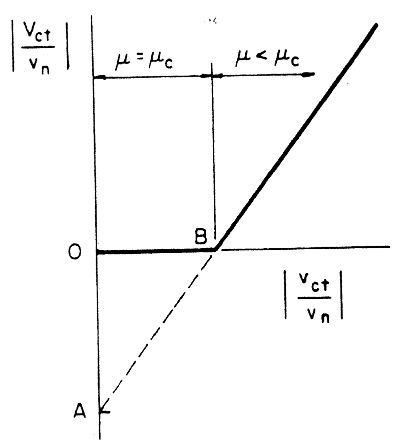

Figure 2. Ratio of the final tangential velocity of the contact point to the initial normal velocity where rolling exists for initial velocity ratios less than point B and sliding for initial velocity ratios greater than point B.

For $\mu = \mu_c$, the tangential impulse is large enough to cause sliding to stop before separation. This is the line from point O to point B and corresponds to rolling at separation. For $|\mu| < |\mu_c|$, relative tangential motion continues through separation and corresponds to the solid line with a unit slope beginning at point B. Point A, the intercept, can be used to determine the value of μ from experimental data, providing that e is known. This intercept can be identified from Equation (18).

3 Planar Impact With Tangential Restitution

The case is now considered where elastic energy can be stored in a mode which can cause the tangential velocity v_{ct} to reverse itself following an impact in a manner different than discussed above. To account for this, Equation (2) can be changed to a form analogous to Equation (1). Thus

$$V_{ct} = V_t - d_d\Omega = -e_t(v_t - d_d\omega) \tag{20}$$

where e_t is a tangential coefficient of restitution. To complete the analogy, e_t is bounded in a matter similar to e, that is, $0 \le e_t \le 1$, although other values can make physical sense as will be seen shortly. The system equations (Equations (1), (20) and (3)) remain linear in the final velocities and again can be solved easily. For expediency, the equations for $d_c = 0$ and $d_d = \rho$ are examined. These are

$$V_n = v_n - (1 + e)\, v_n \tag{21}$$

$$V_t = v_t - \frac{1}{1 + \lambda}(1 + e_t)(v_t - \rho\omega) \tag{22}$$

and

$$\Omega = \omega + \frac{1}{\rho}\frac{\lambda}{1 + \lambda}(1 + e)\,(v_t - \rho\omega) \tag{23}$$

Equation (21) is an alternate form of Equation (15) but the other 2 equations differ from before. Though they differ, it is possible to examine an equivalence between the impulse ratio μ and e_t. Equating V_t from Equations (16) and (22) gives

$$e_t = \frac{\mu}{\mu_c} - 1 \tag{24}$$

This equivalence makes intuitive sense since $e_t = 0$ indicates zero final relative tangential velocity which is also the case for $\mu = \mu_c$. Furthermore, it is easy to show that

$$P_t = -\frac{m}{1 + \lambda}(1 + e_t)\,(v_t - \rho\omega) \tag{25}$$

Note that if -1 is placed into Equation (25) for e_t, it indicates that the tangential impulse is zero. This also agrees with Equation (24), since $\mu = 0$ implies $P_t = 0$. Using this equivalence, it is possible to expand the definition of μ and not use e_t at all. Since μ is usually *associated with* friction and e_t with restitution, distinct physical processes, the two coefficients will be retained.

V_{ct} can again be plotted as in Fig. 2, but for the current case of restitution using Equation (20), which is a straight line through the origin with a negative slope. When sliding persists to separation, Equation (18) or Equation (19) still applies. Both are superimposed in Fig. 3 along with the data points from the experiments of Maw, *et al* [1977]. Their experimental data corresponds to a 4 inch diameter steel disc colliding with a fixed steel barrier at various angles and speeds. The coefficient of restitution was measured to be $e = 0.93$ and independent sliding experiments provided a coefficient of sliding friction $f = 0.123$. Extending the line with unit slope corresponding to sliding at separation gives an intercept of -0.72; this provides a value of the impulse ratio of $\mu = 0.124$. This seems to indicate that for these experiments, the tangential impulse is generated predominantly through friction, at least when sliding exists throughout the duration of contact. The corresponding experimental tangential restitution coefficient is $e_t = 0.34$. In addition to conducting the experiments, Maw, *et al* [1981], report on an analysis of the problem of a sphere impacting on elastic half space. Their analysis provides a model which fits the data about as well as the bilinear model presented here. Their elastic half space model is not discussed here.

Fig. 3 shows two lines, one from the impact model based on μ and the other based on e_t. Together they can be viewed as a bilinear model of the impact process which, by Fig. 3, seems to do an adequate job of representing the data. In general, if the body has considerable initial spin or approaches the barrier at a low angle of incidence, sliding continues throughout the contact duration. On the other hand, for a given set of initial and physical conditions, typically characterized by $f > |\mu_c|$, the final state of the body will be rolling. If the tangential elastic properties are appropriate and the final conditions are in the rolling range, the tangential velocity of the contact region can reverse as demonstrated by the Maw data. Note that data continues to follow the sliding line until it intersects the line dictated by restitution.

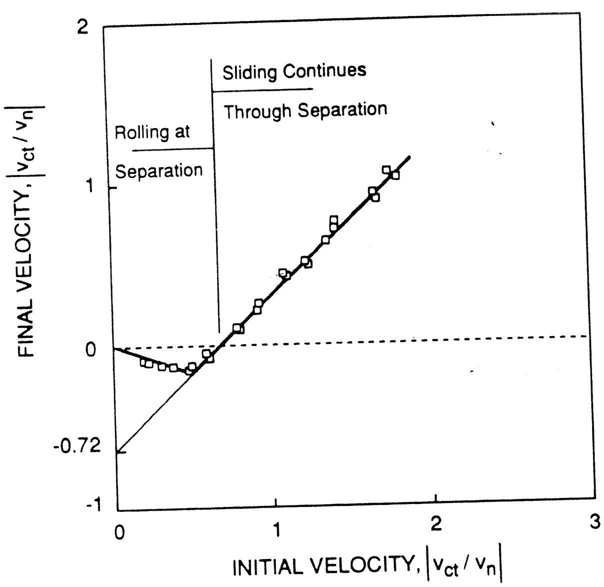

Figure 3. Ratio of the final tangential velocity of the contact point to the initial normal velocity given by the analytical model (heavy line) and the experimental data points of Maw, Barber and Fawcett [1977].

This can be summarized in the following way. For a rigid body impact with

- friction coefficient $f < |\mu_c|$,
- initial velocity ratio r
- coefficients of normal restitution e
- coefficient of tangential restitution e_t

then $V_{ct} = V_i$ $(i = 1 \text{ or } 2,)$ where

$$
\begin{aligned}
V_1 &= v_{ct}\left(1 - \frac{f}{\mu_c}\right), & 0 \le |v_{ct}| \le f\,\frac{1+e}{1+e_t}|v_n|\,(1+\lambda) \\
V_2 &= -\,e_t\, v_{ct}, & f\,\frac{1+e}{1+e_t}|v_n|\,(1+\lambda) \le |v_{ct}|
\end{aligned}
\qquad (26)
$$

If $e_t = 0$, that is, no tangential restitution, and $f > |\mu_c|$, then $V_1 = V_2 = 0$ since the impulse ratio, μ, is equal to μ_c and not f. It is apparent that the tangential impulse (and force) associated with shearing and restitution is transmitted via the frictional impulse (and force). Consequently, for low friction, sliding continues throughout the contact duration and little restitution can occur. For high friction the initial sliding ends quickly and the capability of transmitting elastic tangential energy is greater. However, the initial slope of the final velocity curve is e_t, which is bounded above by 1. Consequently, tangential restitution and its associated velocity reversal is typically small. Of course if the mechanism of tangential impulse transmission is other than friction, for example indentation, the potential for velocity reversal is greater. However, little has been done in studying this phenomenon. The energy loss associated with V_1 above is given by the earlier expressions. For V_2, and bodies with $d_c = 0$, such as discs and spheres, and for $\omega = 0$, the energy loss (expressed as a fraction of the maximum energy loss) is shown by Brach [1988] to be

$$T_L^* = (1 - e^2)\sin^2\alpha + \frac{1}{1+\lambda}(1 - e_t^2)\cos^2\alpha \tag{27}$$

where the angle α is the angle of incidence relative to the tangential axis. For $e = e_t = 1$, no energy is lost. In reality, the likelihood of e_t being unity is slight, since tangential restitution is coupled with friction. Despite this, tangential effects can be visibly significant as with elastomeric balls; see, for example, Johnson [1983].

4 Impact With Torsional Restitution

If a resilient spherical object such as a basketball or volleyball, is spun about a vertical axis and dropped onto a rough flat surface, the spin sometimes reverses. The phenomenon is similar to above except that the storage of elastic energy is of a torsional nature. The appropriate equation involving restitution is simply

$$\Omega_n = -e_z\omega_n \tag{28}$$

where Ω_n is the final angular velocity, ω_n is the initial angular velocity and e_z is a torsional coefficient of restitution. If the initial angular velocity about the normal axis is high enough, its final value may not be opposite in sign; in the presence of Coulomb friction, the angular velocity may simply decrease. Some torsional restitution may still be present even when rotational sliding does not end prior to separation. When it is, its effect is to decrease the rotational velocity somewhat more than friction alone. The corresponding equation of angular impulse and momentum is

$$I_n(\Omega_n - \omega_n) = M_n = \mu_z \rho_p P_n - e_z I_n \omega_n \tag{29}$$

where M_n is the torsional impulse, μ_z is a dimensionless frictional impulse ratio and ρ_p is a pitch radius of the torsional frictional impulse. The second term on the right hand side is the restitutional effect where it is expected that e_z in practice will be small compared to 1. Eliminating P_n and solving for Ω_n gives

$$\Omega_n = (1-e_z)\,\omega_n - \mu_z \frac{m\rho_p}{I_n}(1+e)\,v_n \tag{30}$$

This equation is valid for large values of initial angular velocity when frictional sliding provides that $\mu_z = \pm f$, that is, f is less than a critical μ_z.

A bilinear model results as it did in the tangential problem covered in the previous section. Equation (28) is valid for low values of initial angular velocity and Equation (30) covers higher values. The transition is at a value of ω_n such that

$$\omega_n = \mu_z \frac{m\rho_p}{I_n}(1+e)\,v_n \tag{31}$$

Fig. 4 shows Equations (28) and (30) plotted with the data of Horak corresponding to a rubber ball bouncing from a marble slab. If the effect of torsional restitution did not extend into the range where spinning/sliding continues throughout the contact duration, the last term in Equation (29) would be absent and the first term of Equation (30) would be ω_n. This would give Equation (30) a slope of unity rather than $1-e_z$. The dashed line in Fig. 4 is from Horak [1948] and represents an asymptote of his data. Consequently Equation (30) in the above form fits the data better with a slope of $1-e_z$, but for very large values of ω_n may be better with e_z set to 0. The model agrees fairly well with the experimental data. Horak [1948] develops an elastic model which agrees somewhat better with his experimental data. The reader is referred to that work for details.

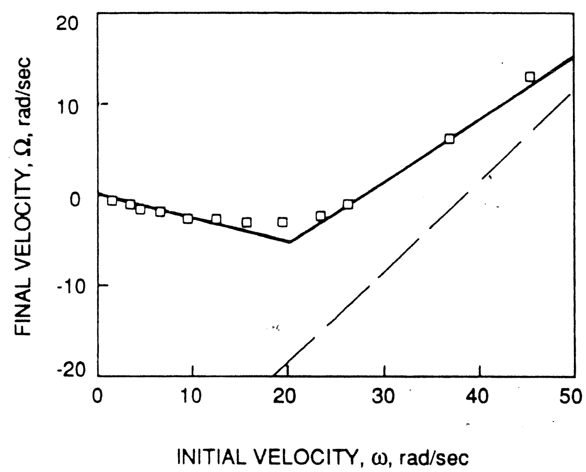

Figure 4 Final angular velocity of a dropped spinning sphere; bilinear theoretical model with experimental data points of Horak (1948). The dashed line is Horak's asymptote with a unit slope.

5 Conclusions

The experimental data of Maw, *et al* [1981] and Horak [1948] demonstrate that the process of restitution is not restricted to normal deformation alone during impacts. The concept of a coefficient of restitution serves as an artifice for energy loss and its use can be extended to cover tangential and torsional deformation.

The bilinear models agree well with the available experimental data. Tangential and torsional velocity reversals depend on friction for transmission of forces and torques. Because of the energy loss associated with the friction the values of the tangential and torsional coefficients could not be expected to approach the value of 1.

6 APPENDIX: Classical Normal Coefficients And Energy Loss

When the kinetic coefficient of restitution is used, an impact analysis must be carried out for each of two phases, or intervals, of the contact duration. This is demonstrated in this Appendix to reveal the relationship between the kinetic coefficient, R, and the classical coefficient e. An impact of a rigid body against a massive surface is considered for simplicity. The following analysis could be carried out for a collision of two rigid bodies but the simpler case provides the necessary insight into the relationships. The partitioning of energy loss between normal and tangential processes is also examined for the case of the kinematic coefficient.

To carry out this analysis, the impact is broken conceptually into two phases. The first phase is when the deformation in the contact region, normal to a common tangent plane, consists primarily of compression (elastic energy storage) and is referred to as the approach phase. The second phase is when some fraction of the stored elastic energy is released and is referred to as the rebound phase. The corresponding time intervals are τ_1 to $\bar{\tau}$ and $\bar{\tau}$ to τ_2. The times τ_1 and τ_2 correspond to the times of initial and final contact. If separation does not occur, the latter is the time at the end of the normal impulse. In the following analysis overbars indicate variable values at time $\bar{\tau}$. Subscripts and superscripts, A and R, refer to approach and rebound phases respectively. Capital velocity variables represent final values at $\tau = \tau_2$ and small, or lower case, velocity values correspond to $\tau = \tau_1$.

For the full contact duration of the body in Fig. 1, Newton's laws in the form of linear and angular momentum give

$$m(V_n - v_n) = P_n \qquad \text{(A-1)}$$

$$m(V_t - v_t) = P_t \qquad \text{(A-2)}$$

$$I(\Omega - \omega) = d_c P_n - d_d P_t \qquad \text{(A-3)}$$

For the approach phase, the corresponding equations are:

$$m(\bar{v}_n - v_n) = P_n^A \qquad \text{(A-4)}$$

$$m(\bar{v}_t - v_t) = P_t^A \qquad \text{(A-5)}$$

$$I(\bar{\omega} - \omega) = d_c P_n^A - d_d P_t^A \qquad \text{(A-6)}$$

Likewise, for the rebound phase

$$m(V_n - \bar{v}_n) = P_n^R \qquad \text{(A-7)}$$

$$m(V_t - \bar{v}_t) = P_t^R \qquad \text{(A-8)}$$

$$I\,(\Omega - \bar{\omega}) = d_c\,P_n^R - d_d\,P_t^R \qquad \text{(A-9)}$$

By definition of the events at time $\bar{\tau}$, the normal velocity of the contact point must be zero. This is

$$\bar{v}_n + d_c\,\bar{\omega} = 0 \qquad \text{(A–10)}$$

Partial impulse ratios must be defined. These are

$$\mu_A = P_t^A / P_n^A \qquad \text{(A-11)}$$

$$\mu_R = P_t^R / P_n^R \qquad \text{(A-12)}$$

where these are simply the ratios of whatever impulses develop for the given physical system and initial conditions. For the solution of the impact problem, they are assumed to be constants which may or may not be known *a priori*.

Omitting some algebraic steps, the following two equations can be derived from the above equations:

$$P_n^R\left(1 + \frac{md_c(d_c - \mu_R d_d)}{I}\right) = m\,(V_n + d_c\Omega) \qquad \text{(A-13)}$$

and

$$P_n^A\left(1 + \frac{md_c(d_c - \mu_A d_d)}{I}\right) = -\,m(v_n + d_c\omega) \qquad \text{(A-14)}$$

Division of Equation (A-13) by Equation (A-14) provides

$$\frac{P_n^R}{P_n^A}\,\frac{I + md_c(d_c - \mu_R d_d)}{I + md_c(d_c - \mu_A d_d)} = -\,\frac{V_n + d_c\Omega}{v_n + d_c\omega} \qquad \text{(A-15)}$$

From Christie [1964], the kinetic coefficient of restitution R is the ratio of impulses on the left hand side of Equation (A-15). From Pestel and Thomson [1968] the kinematic coefficient of restitution e is defined as the right hand side of (A-15). Consequently,

$$R = e\,\frac{I + md_c(d_c - \mu_A d_d)}{I + md_c(d_c - \mu_R d_d)} = e\,\frac{k^2 + d_c(d_c - \mu_A\,d_d)}{k^2 + d_c(d_c - \mu_R\,d_d)} \qquad \text{(A-16)}$$

A primary reason for introducing the coefficient of restitution e is to represent energy loss due to the action of normal forces in the contact region. Energy can also be lost through the work of tangential forces. The interaction of the coefficients e and μ in controlling energy loss is illustrated conveniently by examining the normalized energy loss, T_L^*. According to Lord Kelvin and P.C. Tait [1903] the work of an impulse is the scalar product of the impulse and the average velocity at its point of application. From this the energy loss associated with the normal and tangential impulse components P_n and P_t can be found. This gives

$$T_L^* = \frac{T_L}{T_M} = \frac{1-e^2}{1+\mu_M r}\frac{D_M}{D} + \mu\frac{r(1+e)}{1+\mu_M r}\frac{D_M}{D}\left(1+\frac{V_{ct}}{v_{ct}}\right) \tag{A-17}$$

T_M is the maximum energy loss which is found by letting $e = 0$ and $\mu = \mu_M$. $D_M = D(\mu_M)$ and μ_M is the impulse ratio corresponding to $V_{ct} = 0$ (and $e = 0$). For a point impact, T_M corresponds to the final condition where the body attaches itself and consists of pure rotation about point C with angular velocity Ω. Both terms depend on e and μ. The first term of Equation (A-17) is the energy loss associated with P_n and is controlled by the factor $1-e^2$. The second term is associated with P_t and is controlled by the impulse ratio μ. With the bounds $0 \le e \le 1$ and $0 \le |\mu| \le |\mu_c|$, the energy loss is bounded by

$$0 \le T_L^* \le 1, \tag{A-18}$$

Consequently the kinematic coefficient of restitution e along with the impulse ratio μ serve as valid and complete energy loss parameters. According to W. J. Stronge [1990], Equation (A-17) does not apply to noncentral collisions of rigid bodies for certain conditions of sliding reversal under a Coulomb friction model.

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Nomenclature

A	Sub or superscript indicating the approach phase of an impact
C	Contact point
D	Nondimensional inertial distance (See after Equation 7).
d	Distance
e	Kinematically defined coefficient of restitution
f	Coefficient of friction
I	Centroidal mass moment of inertia
k	Centroidal radius of gyration
M	Impulse of a moment
m	Mass
n	Subscript indicating the normal direction
P	Impulse of a force
R	Sub or superscript indicating the rebound phase of an impact
R	Kinetically defined coefficient of restitution
r	Ratio of initial velocity components (See Eq. 8)
T	Kinetic energy loss
t	Subscript indicating the tangential direction
V,v	Final and initial velocities, respectively
α	Angle of incidence of the contact point; $\tan^{-1}(1/r)$
λ	Ratio of radius of gyration squared to radius squared
μ	Ratio of impulse components, P_t / P_n
ρ	Radius
τ	Time
Ω,ω	Final and initial angular velocities, respectively

Computational Aspect of Contact, Impact and Penetration

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