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CLASSICAL PLANAR IMPACT THEORY AND THE TIP IMPACT OF A SLENDER ROD

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Summary—This paper covers the topic of the planar eccentric impact of a rigid body at a point. The tip impact of a slender rod against a massive surface or barrier is used as a means to illustrate the principles and solution techniques. Two issues are addressed; the first is how the planar impact problem should be, or at least can be, formulated and solved. The second is how close classical solutions match those obtained by methods used in the field of shock and vibration. In formulating the classical impact problem, three well-defined coefficients of restitution are available, the kinematic (defined as a ratio of velocities), kinetic (defined as a ratio of impulses) and energetic (defined as a ratio of energies). Their relationship, advantages and disadvantages are discussed. The best approach to treat tangential impulses in general has not yet reached agreement. For eccentric impacts with Coulomb friction, for example, various combinations of sliding and sticking including tangential velocity reversals can occur during contact. Several approaches have been proposed and all are related. A method using the impulse ratio as a basic parameter is covered and provides solutions for arbitrarily shaped lamina and arbitrary initial conditions. Solutions of the tip impact of a long slender rod using the classical approach are found for various combinations of coefficients of friction and restitution and initial velocities. The final velocities, impulses and energy losses from these solutions are compared with solutions obtained by integrating the differential equations of motion of a rigid rod striking a viscoelastic surface. The comparisons show excellent agreement for most of the cases considered.

1. INTRODUCTION

Some seemingly unusual results have arisen in the search for a solution technique for the eccentric, point collision problem of two rigid bodies (lamina) and the special case of the collision of a rigid body with a massive plane. One author, Stronge [1], refers to *paradoxes* in planar impacts. Mason and Wang [2] claim that *inconsistencies* exist in planar rigid body dynamics when dealing with contact of a moving rigid body against a plane. Though analytical solution techniques to problems such as these have been somewhat elusive, all solutions must follow Newton's laws of motion and should possess characteristics observable in nature. As Stronge [3] points out, central collision problems pose no difficulties since tangential contact velocity reversals cannot occur. Existing methods have produced solutions which show an increase in kinetic energy in eccentric collisions. Examples are given by Kane and Levinson [4] for a double pendulum. Brach [5] points out that energy gains can occur through improper treatment of the tangential impulse. Stronge [3] shows that improper treatment of restitution can also be a cause of energy gains; specifically he has shown that the kinematic coefficient of restitution is dependent on friction if the direction of the force varies during contact.

One of the complexities involves restitution (normal to the contact plane) since three definitions exist giving three distinct coefficients for eccentric impacts [6]. These are the kinematic coefficient, e (defined as a ratio of velocities, also known as Newton's coefficient), the kinetic coefficient, R (defined as a ratio of impulses, also known as Poisson's coefficient) and an energetic coefficient, E^2 (defined as a ratio of energies, or work). The energetic coefficient has been recently proposed by Stronge [3] and leads to a more consistent theory of impact but its use in the formulation of the impact problem leads to a nonlinear equation. Brach [5,7] uses the kinematic coefficient whereas more recently Wang and Mason [8] favor the kinetic. Strong's energetic coefficient is based on directly relating restitution to nonfrictional sources of energy dissipation in a nonconservative system and is important for at least two reasons. Based on its definition, it can be stated that $0 \leqslant E^2 \leqslant 1$ and the coefficient E^2 is independent of the tangential impulse. In contrast, simple numerical bounds for the other two coefficients do not always exist, and e and R generally are not independent of the tangential impulse (typically due to friction). For central impacts, their mutual interdependence is given by $E^2 = eR$ and explains the notation using the square of E. In general their relationship is more complicated [9]. So which should be used? Actually, all of the coefficients have utility, advantages and disadvantages. The classical impact problem is algebraic. When used in the problem formulation, the kinematic (or kinetic) coefficient leads to a linear system of equations for the impact problem whereas the energetic coefficient leads to a nonlinear equation. On the other hand, the energetic coefficient has the significant advantages mentioned above. Since the coefficients are related constants in the impact problem (constants, in the sense that they do not depend on the unknowns, the final velocities), their choice is a matter of convenience. The method covered in this paper attempts to exploit the advantages of both e and E^2.

A second reason that solutions of the planar collision problem can be unrealistic is because of the improper treatment of the tangential impulse. Tangential impulses arise from different physical effects. Examples are indentation, viscous friction and Coulomb (dry) friction. The latter is the only one considered explicitly in this paper and can lead to stick–slip motion. An improper value of the tangential impulse for a given physical system and initial conditions can lead to unrealistic solutions that show a gain in kinetic energy. Different approaches to properly determine the tangential impulse have been used. Routh proposed a graphical procedure which tracks the normal and tangential impulse components, $p_n(\tau)$ and $p_t(\tau)$, (where τ is time) throughout the contact duration. Stronge [1,3] uses an analytical approach to track the normal and tangential impulse components through the impact to reach a solution. Brach [5,7] uses the concept of the ratio μ of the final tangential to normal impulse components. Features of this approach include its applicability to arbitrary tangential processes (not just dry friction) and the ability to express the solution of the impact problem and the corresponding energy loss in terms of the initial velocities and the parameters e and μ. Although Brach has provided bounds on the final or global value of the ratio for classes of solutions, no specific means have been proposed for the calculation of the value of μ for a specific problem or set of initial conditions. This is done in this paper, however. In light of the newly defined energetic coefficient of Stronge [3], a re-evaluation of these procedures is necessary. This is done here for the method based on the impulse ratio and using the kinematic coefficient of restitution. Wang and Mason [8] use Routh's method with the kinetic coefficient of Poisson. There is no inherent problem with this approach except that their solution technique does not take into account the unknown bounds of R and so is incomplete.

Some guidelines are presented in order to make the solution process systematic. Since different possible tangential motions must be considered, some bookkeeping is usually necessary. Additional study is necessary to develop a simple and convenient computational scheme.

Once the solution technique is covered, another question is addressed. That is, how accurate are the predictions of classical impact theory? This question for rigid body collisions has not been approached to any great extent in the past, perhaps because the solution of the eccentric collision problem has not been readily available, as indicated earlier. The question of accuracy is studied comparatively through an example, that of the impact of the tip of a long slender rod against a massive flat surface. This example has been covered in many papers and books on impact [10,11,3,8,9,5] with varying degrees of success at achieving a solution. No experimental results are available for this example. So the results using classical theory are compared to the response obtained from an analytical method commonly used in the field of shock and vibration. Specifically the rigid body response of a rod whose tip strikes a relatively soft, viscoelastic surface is calculated using numerical integration of the vibroimpact model of Hunt and Crossley [12]. The circumstances of a rigid rod hitting a soft surface (a metal rod striking an elastomer, for example) rather than a hard surface were chosen for a specific reason. The soft surface creates conditions leading to relatively long contact times compared to, say, metal-to-metal impact. This in turn allows the rod to acquire a significant change in angular position during the contact interval, a condition violating the assumptions of classical theory. The numerical integration of the rigid body equations calculates position changes. Thus, a rigid rod striking a soft surface should provide a test for classical theory that is more stringent than the impact of a hard rod with a hard surface.

2. CLASSICAL IMPACT THEORY

An arbitrarily shaped rigid body (planar lamina) is shown colliding with a surface at point C in Fig. 1 where the t axis lies on the surface. The classical problem is defined here such that the mass, m, centroidal inertia, mk^2, dimensions, orientation of the body and initial velocities are specified. The impact duration begins at τ_1 and ends at τ_2, where τ is time. A unique solution for the final velocity components and impulse components is assumed to exist and is sought. Three equations for this problem are given by Newton's law as

$$m(V_n - v_n) = P_n \tag{1}$$

$$m(V_t - v_t) = P_t \tag{2}$$

$$mk^2(\Omega - \omega) = yP_n - xP_t. \tag{3}$$

In these equations, capital symbols indicate final values of velocity components V_n, V_t and Ω and impulse components P_n and P_t at the time of separation from the surface at τ_2. Small or lower case symbols v_n, v_t and ω indicate initial values, at time τ_1. The velocity components V_n, V_t, v_n and v_t are mass center velocities. Additional equations are needed since there are five unknowns, V_n, V_t, Ω, P_n and P_t. Based on the assumption of a unique solution, the two constants e and μ are defined such that $e = -V_{Cn}/v_{Cn}$ and $\mu = P_t/P_n$. V_{Cn} and v_{Cn} are normal velocity components at the contact point C. The quantities e and μ are called the kinematic coefficient of restitution and the impulse ratio, respectively, and furnish the remaining two equations

$$V_{Cn} = -ev_{Cn} \tag{4}$$

$$P_t = \mu P_n. \tag{5}$$

These are linear algebraic equations leading to final velocities given by the equations

$$V_n = v_n - (1 + e)(v_n + y\omega)/D \tag{6}$$

$$V_t = v_t - \mu(1 + e)(v_n + y\omega)/D, \quad |\mu| \leqslant |\mu_c| \tag{7}$$

$$\Omega = \omega - (y - \mu x)(1 + e)(v_n + y\omega)/Dk^2, \quad |\mu| \leqslant |\mu_c| \tag{8}$$

where $D = 1 + y(y - \mu x)/k^2$ and μ_c represents a characteristic value of the impulse ratio as discussed by Brach [7]. Equation (4) is equivalent to an equation expressing energy

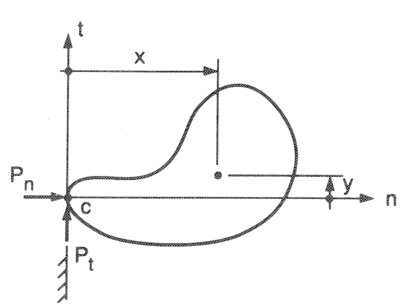

FIG. 1. Free body diagram of a lamina with mass, m, and centroidal inertia, mk^2.

loss which itself is quadratic in the velocities. It can be shown that as a consequence, two solutions exist. One is for $+e$ and the other for $-e$. A negative value of e implies penetration rather than rebound. The interpretation and utility of this solution is not discussed here. With an added constraint, $e > 0$, the solution of Eqns (1)–(5) is unique.

Coefficients of restitution

Following the introduction of the energetic coefficient of restitution, E, by Stronge [3], Brach [6] has shown that for planar collisions

$$e = R\frac{k^2 + y(y - \mu_R x)}{k^2 + y(y - \mu_A x)}, \quad 0 \leqslant E^2 \leqslant 1. \tag{9}$$

The quantities μ_A and μ_R in Eqn (9) are the ratios of the tangential to normal impulses over the approach and rebound intervals, respectively, and are called partial impulse ratios. The kinetic coefficient of restitution is defined by $R = P_n^R / P_n^A$ where these impulses are the normal rebound and approach impulses, respectively. Equation (9) shows that when $y = 0$ (the condition of a central impact) e and R are identical. This also follows when $\mu_A = \mu_R$, particularly for the frictionless case where both are zero. A relationship between e and E^2 similar in nature to Eqn (9) can be found, but it takes multiple forms due to the presence of stick–slip motion. For a discussion of this topic see Smith [9].

Impulse ratios

Because impulse ratios are used in the approach followed in this paper it is necessary to investigate their behavior and how they are computed. Consider an arbitrary subinterval of time $\Delta\tau = \tau_b - \tau_a$ within the contact duration, that is $\tau_1 \leqslant \tau_a \leqslant \tau_b \leqslant \tau_2$. For this interval, equations comparable to Eqns (1), (2) and (3) are

$$m[v_n(\tau_b) - v_n(\tau_a)] = p_n(\tau_b) - p_n(\tau_a) \equiv \Delta p_n \tag{10}$$

$$m[v_t(\tau_b) - v_t(\tau_a)] = p_t(\tau_b) - p_t(\tau_a) \equiv \Delta p_t \tag{11}$$

$$mk^2[\omega(\tau_b) - \omega(\tau_a)] = y\Delta p_n - x\Delta p_t \tag{12}$$

where $p_n(\tau)$ and $p_t(\tau)$ are dynamic impulse variables defined over an indefinite time interval τ–τ_1. Instantaneous velocities at the contact point can be written as

$$v_{Cn}(\tau) = v_n(\tau) + y\omega(\tau) \tag{13}$$

$$v_{Ct}(\tau) = v_t(\tau) - x\omega(\tau). \tag{14}$$

Equations (10), (11) and (12) can be solved for the velocities $v_n(\tau_b)$, $v_t(\tau_b)$ and $\omega(\tau_b)$ and substituted into Eqns (13) and (14) evaluated at $\tau = \tau_b$. The resulting pair of equations are linear in the impulse variables Δp_n and Δp_t. Solving them for Δp_n and Δp_t and taking the ratio gives

$$\mu(\Delta\tau) = \frac{\Delta p_t}{\Delta p_n} = \frac{(k^2 + y^2)[v_{Ct}(\tau_b) - v_{Ct}(\tau_a)] + xy[v_{Cn}(\tau_b) - v_{Cn}(\tau_a)]}{(k^2 + x^2)[v_{Cn}(\tau_b) - v_{Cn}(\tau_a)] + xy[v_{Ct}(\tau_b) - v_{Ct}(\tau_a)]}. \tag{15}$$

This equation can be used to determine the impulse ratio over any interval τ_b–τ_a. The technique illustrated here involves solutions of Eqn (15) over sequential subintervals of unidirectional sliding or no sliding. In some solutions, $\mu(\Delta\tau)$ is known and Eqn (15) is solved for a velocity; in others, $\mu(\Delta\tau)$ is found from known velocities. This procedure furnishes the various impulse ratios encountered in the above solution equations for the classical impact problem and eventually provides all of the unknowns.

Examples

As an example, consider the tip collision of a slender rod. The problem is illustrated in Fig. 2; Table 1 lists the specific conditions chosen. Coulomb friction with a coefficient of $f = 0.01$ generates the tangential force and impulse. Using Eqns (6), (7) and (8), the final tangential contact velocity is $V_{Ct} = v_t - 3(1 + e)/5$ when the surface is frictionless, that is,

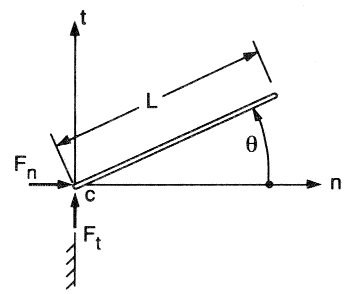

FIG. 2. Free body diagram of a slender rigid rod striking a massive plane at point C.

TABLE 1. SLENDER ROD EXAMPLE, PHYSICAL PARAMETERS

Mass	$m = 1$ (kg)
Length	$L = 1$ (m)
Moment of inertia	$I = mk^2$ (kg m^2)
Radius of gyration	$k^2 = L^2/12$ (m^2)
Orientation	$\theta = 45°$
Centroidal coordinates	$x = y = L/2\sqrt{2}$ (m)
Initial velocities	$v_n = -1, \omega = 0$ (rad s^{-1})
	$v_t = -1.0, 0, 0.2, 0.6$ (m s^{-1})
Coefficient of restitution	$E^2 = 1.0$
Coefficient of friction	$f = 0.01$

TABLE 2. SEQUENCE OF EVALUATIONS OF EQN (15)

Time interval, $\Delta\tau$	Velocity changes	Impulse ratio $\mu(\Delta\tau)$	Computed quantity
1. $\tau_1 \to \tau_{SR}^*$	$(v_{Cn}, v_{Ct}) \to (v_{Cn}(\tau_{SR}), 0)$	$-f$	$v_{Cn}(\tau_{SR}) = -0.010$
2. $\tau_{SR} \to \bar{\tau}^{**}$	$(v_{Cn}(\tau_{SR}), 0) \to (0, v_{Ct}(\bar{\tau}))$	$+f$	$v_{Ct}(\bar{\tau}) = -0.006$
3. $\tau_1 \to \bar{\tau}$	$(v_{Cn}, v_{Ct}) \to (0, v_{Ct}(\bar{\tau}))$	?	$\mu_A = -9.788 \times 10^{-3}$
4. $\bar{\tau} \to \tau_2$	$(0, v_{Ct}(\bar{\tau})) \to (V_{Cn}, V_{Ct})$	$\mu_R = +f$	$V_{Ct}(E, f) = -0.596$
5. $\tau_1 \to \tau_2$	$(v_{Cn}, v_{Ct}) \to (V_{Cn}, V_{Ct})$	?	$\mu = 1.3564 \times 10^{-4}$

* Stop and reverse time.
** Time separating approach and rebound.
$v_n = -1.0$, $v_t = 0.6$ m s^{-1}, $E^2 = 1.0$, $f = 0.01$.

for $P_t = \mu = f = 0$. Consider first the case where $v_{Ct} = v_t = 0.06$ m s^{-1}. Brach [7] shows that for a kinematic coefficient value of $e = 1$, the only realistic solution is for the value of $\mu = 0$ (and, coincidentally, $f = 0$). For the small value of friction used here, it would be expected that a reversal of tangential tip velocity should occur as in the frictionless case that that it would be achieved for $e < 1$. Table 2 outlines a sequence of evaluations of Eqn (15) which leads to the final impulse ratio, $\mu = P_t/P_n$.

The first evaluation of Eqn (15) is from the beginning of contact at τ_1 to the time τ_{SR} at which time sliding stops and reverses. For this interval, sliding is unidirectional and $\mu(\Delta\tau) = -f$. Equation (15) is solved for the value of the normal contact velocity and gives $v_{Cn}(\tau_{SR}) = -0.010$. Because of its negative value, this shows that the impact is still in the approach phase. The next evaluation is over the interval from τ_{SR} to $\bar{\tau}$, the latter of which is defined as the time approach ends and rebound begins, and is when $v_{Cn}(\bar{\tau}) = 0$. During this interval $\mu(\Delta\tau) = +f$ since sliding has reversed. Using Eqn (15) again provides a value of the tangential velocity of $v_{Ct}(\bar{\tau}) = -0.006$. The next evaluation listed in Table 2 is for the interval of $\tau_1 - \bar{\tau}$ and gives a value of the approach impulse ratio $\mu_A = -9.788 \times 10^{-3}$.

The next interval is from $\bar{\tau}$ to τ_2 which is the rebound phase so $\mu(\Delta\tau) = \mu_R = +f$. This provides a value of the final tangential velocity at the contact point of $V_{Ct}(\tau_2) = -0.596$. The fifth and last evaluation of Eqn (15) in Table 2 is over the entire interval of contact and provides the final impulse ratio $\mu = \mu(\tau_2) = 1.3564 \times 10^{-4}$.

The coefficient of restitution did not enter into the first three evaluations of Eqn (15). Evaluations 4 and 5 required the use of $V_{Cn} = -ev_{Cn}$. Since μ_A and μ_R are both known at evaluation 4, e can be calculated from a given value of E. In this example, E^2 is chosen to be 1, giving $e = 0.9941$ and $\mu = 1.3564 \times 10^{-4}$. These values can be substituted into Eqns (6), (7) and (8) to calculate the final angular velocity and the final mass center velocities. The final impulses can be calculated from Eqns (1) and (2), providing a complete solution.

Consider the same example but with $v_t = 0.2$. Rather than a specific solution, consider all physically realistic solutions for ranges of the coefficients of restitution and impulse ratios. The kinetic energy loss as a function of μ for $e = 0$, 0.5 and 1 corresponding to the solution equations (6), (7) and (8) is shown in Fig. 3. If one considers solutions which gain kinetic energy as being unrealistic, then not all combinations of e and μ provide realistic solutions. For example, $e = 1$ and $f = 0.9$ gives $\mu = 0.5319$ and final velocities with a gain in energy of more than 12% (relative to the preimpact energy). A way of insuring realistic solutions is to place bounds on the energy loss curves by limiting e and μ as suggested in Brach [7]. The bounds discussed there for this example are reproduced in Fig. 3 as the curves OC and AA'B and the $e = 1$ and $e = 0$ curves. The A'B curve is the $\mu = \mu_0$ curve, where sliding ends at or before separation. The energy loss curves corresponding to $E = 1$, $E = 0.5$ and $E = 0$, obtained from the solution technique discussed above are added to the curves. Energy loss for $0 \leqslant E \leqslant 1$ fills the space between the $E = 1$, $E = 0$, OC and A'B curves. The energy bound curves suggested by Brach [7] include the space between the $e = 1$, $E = 1$ and AA' curves. Whether these points are physically attainable should likely be answered by experiment. Results from the viscoelastic analysis presented later in this paper suggest, however, that at least some of these points with values of $E > 1$ could be physically attainable. Classical impact theory restricts solutions to $E \leqslant 1$.

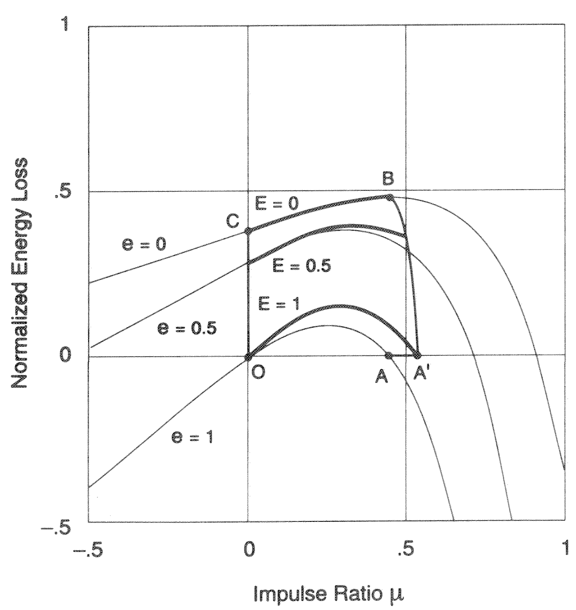

FIG. 3. Kinetic energy lost in a slender rod impact, as a fraction of its initial energy for $v_t = 0.2$. Thin curves correspond to solution equations (7), (8) and (9) for a broad range of values of e and μ. Curves OC, AA' and A'B are bounds established by $\mu = 0$, $\mu = \mu_T$ and $\mu = \mu_0$, respectively. Heavy curves correspond to solution equations (7), (8) and (9) for indicated values of Stronge's energetic coefficient of restitution.

Observations

Some guidelines can be garnered from the above equations and the first example solution. It generally proves helpful to examine the frictionless solution before a solution is sought for nonzero f. This is done easily by setting $\mu = 0$ in Eqns (6), (7) and (8) and serves to indicate the initial direction of sliding and whether or not a reversal is likely to occur. Another factor to keep in mind is that the coefficient of restitution does not enter into the evaluations of Eqn (15) until final velocities are encountered. Solving the problem for additional values of the coefficient of restitution requires only 1 or 2 recalculations. Another observation is that the solution procedure not only determines the final impulse ratio μ, but furnishes the final contact point velocities V_{Cn} and V_{Ct}. This is not a complete solution of the problem, however. Equations (6), (7) and (8) must still be used to determine the angular velocity Ω and the mass center velocities V_n and V_t.

From the example with $v_t = 0.2$, it is seen that energy bounds exist on the impact problem solutions. These bounds are presented in Brach [7] for several initial tangential velocities, but only for the kinematic coefficient, e. The establishment of the energy loss curves based on E is more informative and illustrates the significance of Stronge's energetic coefficient.

3. IMPACT SIMULATION

Consider a completely different approach to the determination of the dynamic response of a rod striking a surface. The mass center coordinates of the rod are $n(\tau)$ and $t(\tau)$ and the angular position is $\theta(\tau)$, where τ represents time. Figure 2 shows a free body diagram illustrating normal and tangential forces $F_n(\tau)$ and $F_t(\tau)$. Newton's laws can be written as

$$m\ddot{n} = F_n \tag{16}$$

$$m\ddot{t} = F_t \tag{17}$$

$$mk^2\ddot{\theta} = F_n L\sin\theta - F_t L\cos\theta. \tag{18}$$

Suppose the rod is perfectly rigid and the surface provides elasticity and dissipation. Hunt and Crossley [12] discuss different types of elastic and viscous models which apply to the process of impact. Along the line of their recommendations, the normal force is chosen here to have the form

$$F_n = \kappa n_C^p - \zeta\kappa\dot{n}_C n_C^p = -\kappa n_C^p(1 + \zeta\dot{n}_C). \tag{19}$$

Here n_C is the normal displacement of the tip of the rod at the contact point, κ is the stiffness of the surface, ζ is a damping factor and the exponent p is a constant. In contrast to the type of linear viscous damping usually used in vibration theory, the damping in Eqn (19) does not have an instantaneous value at the initiation of contact. For Coulomb friction the tangential force can be written as

$$F_t = -fF_n\dot{t}_C/|\dot{t}_C|, \quad |\dot{t}_C| \neq 0 \tag{20}$$

where $\dot{t}_C$ is the tangential tip velocity. Equations (16)–(20) form a system of three ordinary differential equations which are solved numerically using a fourth order Runge–Kutta–Gill method. As part of the solution, the motion and forces must be monitored to determine if relative tangential motion at the tip becomes zero and whether or not it remains zero. When relative tangential motion at C is zero, Eqn (20) is replaced by a kinematic constraint force of the form

$$F_t = \frac{F_n L^2\cos\theta\sin\theta - 2mLk^2\dot{\theta}^2\sin\theta}{4k^2 + L^2\cos^2\theta}, \quad \dot{t}_C = 0. \tag{21}$$

To assure accuracy in estimating the stopping conditions, when the tangential tip velocity changed sign, the zero crossing time was estimated by interpolation, the integration interval was reduced and the last integration was repeated. The condition for "sticking" is if the

TABLE 3. RESULTS FROM SIMULATION AND CLASSICAL SOLUTION* ROD ORIENTED AT 45°

Damping parameter ζ ($\mathrm{s\,m^{-1}}$)	Contact duration $\tau_2 - \tau_1$ (s)	Final position (deg)	Normal impulse P_n (N m)	Impulse ratio μ	Kinematic coeff e	Energy coeff E^2	Contact velocity V_{Ct} ($\mathrm{m\,s^{-1}}$)	Angular velocity Ω ($\mathrm{rad\,s^{-1}}$)	Energy loss T_L (%)
0.00	0.061	50.8	0.774	0.000	1.079	1.054	−0.464	192.9	0.32
0.04	0.061	50.8	0.764	0.000	1.052	1.000	−0.450	190.4	1.81
		45.0	**0.798**	**0.000**	**0.994**	**1.000**	**−0.596**	**193.9**	**0.35**
1.69	0.064	51.1	0.559	−0.004	0.500	0.213	−0.162	138.6	23.6
		45.0	**0.599**	**−0.003**	**0.500**	**0.253**	**−0.303**	**146.0**	**22.1**
20.0	0.023	46.7	0.409	−0.009	0.050	0.002	−0.007	100.8	29.3
		45.0	**0.418**	**−0.009**	**0.050**	**0.003**	**−0.036**	**102.4**	**29.3**

* Classical solution results in bold; common values underlined.
Initial normal velocity $v_n = -1\ \mathrm{m\,s^{-1}}$; initial tangential velocity, $v_t = 0.6\ \mathrm{m\,s^{-1}}$; friction coefficient, $f = 0.010$; surface stiffness, $\kappa = 1000\ \mathrm{N\,m^{-1}}$.

kinematic constraint force given by Eqn (21) is below the friction force, otherwise sliding continues in the reverse direction. The condition for the end of impact is $n_C \geqslant 0$.

The surface elastic parameters such as the exponent $p = 1$ and the stiffness $\kappa = 1000\ \mathrm{N\,m^{-1}}$ were chosen to represent a relatively soft surface. Values of the damping factor ζ are varied as necessary to produce desired coefficients of restitution.

Results

Consider first the example conditions covered earlier for a friction coefficient $f = 0.01$ and an initial tangential velocity of $v_t = 0.6\ \mathrm{m\,s^{-1}}$. Table 3 lists some pertinent results from the simulation and corresponding results from classical impact theory. With no dissipation, $\zeta = 0$, the simulation gives a response with an energetic coefficient $E = 1.027$ and a kinematic coefficient $e = 1.079$. The normal tip speed increases to $1.079\ \mathrm{m\,s^{-1}}$ as a result of the impact. (With zero friction the corresponding values are $E = 1.027$ and $e = 1.085$. Despite values greater than 1, overall energy is conserved. These results are not unrealistic because in the simulation, the angular position changes and permits a final normal tip speed greater than the original.) For $f = 0.01$, the tangential tip speed reverses but ends up lower in magnitude than its original value. The angular velocity, which originally is zero, increases significantly due to the offset of the forces from the mass center. For the remaining data in Table 3, surface dissipation is added (in a trial-and-error fashion) to reach specific values of the coefficient of restitution to form a basis of comparison. In some cases the two methods give noticeably different final (tangential) contact speeds V_{Ct}. The final angular velocities are quite close, however. The kinetic energy losses, T_L, show good agreement, particularly as the damping is increased. The classical solution's assumption of negligible displacements (as viewed by the change in angular position) also becomes better as the damping is increased (and the impact duration becomes shorter).

For the system defined in Table 1, Tables 4 and 6 summarize results from the simulation for four different initial tangential speeds. Tables 5 and 7 show comparisons with results from impact theory. The various values of initial tangential speed were chosen to correspond to the cases studied and presented by Brach [5,7]. In addition to varying the damping, the friction coefficient is increased to reach and exceed various limiting conditions such as when sliding ends just as $\tau = \tau_2$ or earlier during the contact duration (by further increases in friction). When the friction coefficient equals or exceeds a certain limiting value, $V_{Ct} = 0$. For these cases the classical solution can be found using the limiting value $\mu = \mu_0$, [7] where

$$\mu_0 = \frac{(k^2 + y^2)r/(1 + e) + xy}{k^2 + y^2 + xyr/(1 + e)} \tag{22}$$

and $r = (v_t - x\omega)/(v_n + y\omega)$. A review of the results shows that the classical theory predicts these limiting values of the impulse ratio well. Kinetic energy losses agree quite well as do

TABLE 4. SIMULATION RESULTS, SLENDER ROD TIP IMPACT

Friction coeff f	Normal stiffness k ($N m^{-1}$)	Damping parameter ζ ($s m^{-1}$)	Contact duration $\tau_2 - \tau_1$ (s)	Final position (deg)	Normal impulse P_n (N m)	Impulse ratio μ	Kinematic coeff e	Kinetic coeff R	Energy coeff E^2
Initial tangential velocity, $v_t = 0.0$ m s^{-1}									
0.0	1000	0.0	0.061	50.8	0.777	0.0	1.085	0.972	1.054
0.5975*	1000	0.0	0.077	49.7	1.214	0.597	1.070	0.975	1.043
1.0†	1000	0.0	0.077	49.7	1.219	0.597	1.078	0.975	1.051
0.0	1000	1.69	0.064	51.1	0.561	0.0	0.501	0.423	0.212
0.5888‡	1000	1.61	0.080	50.0	0.882	0.589	0.500	0.444	0.222
0.5905*	1000	1.61	0.080	50.0	0.883	0.589	0.503	0.444	0.223
1.0†	1000	1.68	0.080	49.9	0.882	0.583	0.500	0.429	0.214
0.0	1000	20.0	0.023	46.7	0.411	0.0	0.050	0.040	0.002
0.5968*	1000	20.0	0.031	46.5	0.645	0.596	0.050	0.043	0.002
1.0†	1000	20.0	0.030	46.4	0.641	0.545	0.050	0.041	0.002
Initial tangential velocity, $v_t = 0.2$ m s^{-1}									
0.0	1000	0.00	0.061	50.8	0.777	0.000	1.085	0.972	1.054
0.5987*	1000	0.00	0.075	50.4	1.062	0.521	0.971	0.979	0.951
1.0†	1000	0.00	0.075	50.5	1.060	0.522	0.969	0.977	0.947
0.0	1000	1.70	0.064	51.1	0.560	0.000	0.499	0.421	0.210
0.5993‡	1000	1.58	0.078	50.7	0.792	0.483	0.501	0.472	0.236
1.0†	1000	1.58	0.078	50.7	0.791	0.485	0.500	0.472	0.236
0.0	1000	20.0	0.023	46.7	0.411	0.0	0.050	0.040	0.002
0.6019*	1000	20.0	0.029	46.7	0.566	0.463	0.050	0.044	0.002
1.0†	1000	20.0	0.029	46.7	0.567	0.481	0.050	0.044	0.002

* Smallest coefficient of friction to cause zero tangential velocity at separation.
† No sliding at separation.
‡ Smallest coefficient of friction to cause sliding to stop and reverse.
Oriented at 45°; initial normal velocity $v_n = -1$ m s^{-1}.

the angular velocity changes. For some variables the percentage error may be quite large but in most cases this is when the values of the variables are small in magnitude compared to most others in the problem.

A specific aspect of the comparisons worth noting concerns bounds on the impulse ratio, in particular μ_0. With some exceptions, Tables 5 and 7 show that the simulation gives a value of μ_0 within a few percent of that from impact theory. Exceptions occur for large values of e and where $v_t = 0.6$. It appears that when the impact theory solution is a good approximation, μ_0 is a good approximation of the value of the limiting value of friction. Higher values of friction still result with no sliding at separation but still have the same impuse ratio, μ_0 which is predicted by the methods of this paper.

4. DISCUSSION AND CONCLUSIONS

Earlier work on the solution of the classical impact problem by Brach [5,7] shows the existence of bounds on the final values of the impulse ratio $\mu = \mu(\tau_2)$ such as μ_0 given by Eqn (22). Two others are commonly encountered. These are μ_T and μ_{max}, both of which depend on the coefficient of restitution. The first, μ_T, is the value of μ, which if exceeded (in magnitude), leads to an impact with a gain in kinetic energy (and provides the bound AA′ in Fig. 3, for example). The latter is the value of μ which, with $e = 0$ ($E^2 = 0$), leads to the maximum possible energy loss for the given physical system and initial conditions. The latter can be quite useful in applications to determine an upper bound of the energy loss. The former, μ_T, deserves some discussion here. In earlier work, the need for μ_T arose because the problem formulation is based on the kinematic coefficient e and did not include

TABLE 5. COMPARISON OF IMPACT THEORY TO SIMULATION RESULTS, SLENDER ROD TIP IMPACT

Normal impulse P_n (N m)	Impulse ratio μ	Final sliding velocity V_{Ct} ($m s^{-1}$)	Final angular velocity Ω ($deg s^{-1}$)	Energy loss T_L (%)	Normal impulse $P_n(\tau_2)$ (N m)	Impulse ratio μ	Final sliding velocity $\dot{t}_c(\tau_2)$ ($m s^{-1}$)	Final angular velocity $\dot{\theta}(\tau_2)$ ($deg s^{-1}$)	Energy loss T_L (%)
Initial tangential velocity, $v_t = 0.0\ m s^{-1}$									
Coefficient of restitution, $e = 1$									
0.800	0.0	−1.2	194.5	0.0	0.777	0.0	−1.07	193.5	0.00
1.250	0.6*	0.0	121.5	0.0	1.214	0.597	0.0	128.5	0.95
1.250	0.6*	0.0	121.5	0.0	1.219	0.597	0.0	129.0	0.02
Coefficient of restitution, $e = 0.5$									
0.600	0.0	−0.90	145.9	30.0	0.561	0.0	−0.76	138.4	32.1
0.938	0.6*	0.0	91.2	46.9	0.882	0.589	0.0	92.5	49.9
0.938	0.6*	0.0	91.2	46.9	0.883	0.589	0.0	92.6	49.8
0.938	0.6*	0.0	91.2	46.9	0.882	0.583	0.0	92.5	49.9
Coefficient of restitution, $e = 0.05$									
0.420	0.0	−0.63	102.1	39.9	0.411	0.0	−0.60	100.5	39.8
0.656	0.6*	0.0	63.8	62.3	0.645	0.596	0.0	64.0	62.2
0.656	0.6*	0.0	63.8	62.3	0.641	0.545	0.0	64.7	61.4
Initial tangential velocity, $v_t = 0.2\ m s^{-1}$									
Coefficient of restitution, $e = 1.0$									
0.800	0.0	−1.0	194.5	0.0	0.776	0.0	−0.87	193.5	0.0
1.101	0.456†	−0.2	145.6	0.0	1.062	0.521	0.0	135.1	0.8
1.101	0.456†	−0.2	145.6	0.0	1.060	0.522	0.0	135.0	1.0
Coefficient of restitution, $e = 0.5$									
0.600	0.0	−0.7	145.9	28.8	0.560	0.0	−0.56	138.3	30.9
0.862	0.507*	0.0	103.7	33.3	0.792	0.483	0.0	105.0	36.4
0.862	0.507*	0.0	103.7	33.3	0.791	0.485	0.0	105.1	36.4
Coefficient of restitution, $e = 0.05$									
0.420	0.0	−0.43	102.1	38.4	0.411	0.0	−0.40	100.4	38.3
0.580	0.46*	0.0	76.0	47.9	0.566	0.463	0.0	76.1	47.9
0.580	0.46*	0.0	76.0	47.9	0.567	0.481	0.0	75.9	48.1

* Impulse ratio, μ_0.
† Impulse ratio, μ_T.
Oriented at 45°; initial normal velocity $v_n = -1\ m s^{-1}$.

the bounds on E^2. It is possible that values of $e \leqslant 1$ for eccentric impacts can lead to corresponding values of $E^2 > 1$. At the time the bound μ_T was derived, Stronge's energetic coefficient had not yet been introduced into the literature and a relationship between e and E^2 was not available. Presently this relationship is better understood and the bound $E^2 \leqslant 1$ can be used to establish bounds on e and the use of μ_T may no longer be necessary. However it was seen from the simulation that values of $E^2 > 1$ can occur, so this topic needs more study.

The limiting impulse ratio, μ_0, conveniently provides the value, $P_t = \mu_0 P_n$, of the tangential impulse for the classical solution necessary to cause sliding to end at or before the time of separation. This same value also provides partial bounds of the energy loss curves $T_L(E)$ (as shown in Fig. 3) and an overall upper bound on energy loss with $\mu_{max} = \mu_0(e)|_{e=0}$.

Four sets of initial conditions were examined, each differing by the initial tangential velocity. Values of $v_t = 0.0$, 0.2, 0.6 and -1.0 were examined. The differences between results of classical impact theory and the simulation for the first two values, 0.0 and 0.2 are typically quite small. That is, the final velocities, impulses, impulse ratios (final and critical) and energy losses differ typically by about 10% or less; in many cases, the differences are only a few percent. Differences are greatest for some of the impacts with an initial

TABLE 6. SIMULATION RESULTS, SLENDER ROD TIP IMPACT

Friction coeff f	Normal stiffness k ($N m^{-1}$)	Damping parameter ζ ($s m^{-1}$)	Contact duration $\tau_2 - \tau_1$ (s)	Final position (deg)	Normal impulse P_n (N m)	Impulse ratio μ	Kinematic coeff e	Kinetic coeff R	Energy coeff E^2
Initial tangential velocity, $v_t = 0.6 m s^{-1}$									
0.0	1000	0.0	0.061	50.8	0.777	0.0	1.085	0.972	1.054
0.5908*	1000	0.0	0.068	51.2	0.801	0.278	0.822	1.095	0.900
1.0†	1000	0.0	0.069	51.3	0.787	0.282	0.800	1.061	0.848
0.0	1000	1.69	0.064	51.1	0.561	0.0	0.501	0.423	0.212
0.5481‡	1000	1.21	0.069	50.4	0.616	0.166	0.500	0.641	0.320
0.5899*	1000	1.31	0.070	51.5	0.616	0.162	0.500	0.611	0.306
1.0†	1000	1.27	0.070	51.6	0.613	0.159	0.500	0.613	0.307
0.0	1000	20.0	0.023	46.7	0.411	0.0	0.050	0.040	0.002
0.5950*	1000	20.0	0.027	47.2	0.406	−0.014	0.050	0.051	0.003
1.0†	1000	20.0	0.028	47.3	0.405	−0.017	0.050	0.051	0.003
Initial tangential velocity, $v_t = -1.0 m s^{-1}$									
0.0	1000	0.00	0.061	50.8	0.777	0.000	1.085	0.972	1.054
0.8555*	1000	0.00	0.089	47.7	1.617	0.855	1.039	0.985	1.023
1.0†	1000	0.00	0.089	46.9	1.785	0.813	1.291	0.789	1.018
0.0	1000	1.69	0.064	51.1	0.561	0.000	0.501	0.423	0.212
0.9120*	1000	1.49	0.097	46.9	1.301	0.911	0.499	0.480	0.240
1.0†	1000	1.77	0.096	46.3	1.307	0.907	0.500	0.308	0.154
0.0	1000	20.0	0.023	46.7	0.411	0.0	0.050	0.040	0.002
0.9882*	1000	20.0	0.071	45.2	1.031	0.988	0.050	0.050	0.003
1.0†	1000	20.0	0.055	45.1	1.031	0.989	0.050	0.031	0.002

* Smallest coefficient of friction to cause zero tangential velocity at separation.
† No sliding at separation.
‡ Smallest coefficient of friction to cause sliding to stop and reverse.
Oriented at 45°; initial normal velocity $v_n = 1 m s^{-1}$.

velocity of $v_t = 0.6$. For example, for $e = 1$ and a friction coefficient $f = 1.0$, the final impulse ratio is 0.0 from classical theory but is 0.282 from the simulation; correspondingly the energy loss is 0.0 from classical theory, but 6.94% from the simulation. Overall, column-to-column comparisons in Tables 5 and 7 show that the classical method and simulation agree quite well. However, there are notable exceptions. Unfortunately, there seem to be no clear trends which lead to a general criterion on accuracy.

In general, the procedure of sequential applications of Eqn (15) for $\mu(\Delta\tau)$ and use of Eqns (6), (7) and (8) leads to a solution of the classical impact problem. Examples given include contact point velocity reversals which are usually the most tedious cases to solve. (The reader may wish to take note that a spreadsheet can be organized to conveniently handle solutions of these problems.) Although the examples here are for a single rigid body colliding with a barrier, the procedure is valid for the relative velocities of the impact of two rigid bodies. Once μ is found, the final solution equations for a two-mass impact can then be used to complete the solution. Of course, solutions of the impact equations for a class or range of friction values can still be found using the solution equations and bounds on μ such as shown in Fig. 3 without the use of Eqn (15). This approach is particularly useful for cases where Coulomb friction is not appropriate and where methods based on integration are impractical such as for the impact of highway vehicles [13].

Mason and Wang [2] have approached the problem of a rod with an established tip contact (zero initial normal velocity) sliding with constant tangential velocity along a flat surface in the presence of Coulomb friction. They conclude that the only solution has the nature of an impact with the rod bounding from the surface. They further claim that their solution demonstrates an inconsistency in Newtonian mechanics. Although this particular

TABLE 7. COMPARISON OF IMPACT THEORY TO SIMULATION RESULTS, SLENDER ROD TIP IMPACT

Normal impulse P_n (N m)	Impulse ratio μ	Final sliding velocity V_{Ct} (m s^{-1})	Final angular velocity Ω (deg s^{-1})	Energy loss T_L (%)	Normal impulse $P_n(\tau_2)$ (N m)	Impulse ratio μ	Final sliding velocity $\dot{t}_c(\tau_2)$ (m s^{-1})	Final angular velocity $\dot{\theta}(\tau_2)$ (deg s^{-1})	Energy loss T_L (%)
Initial tangential velocity, $v_t = 0.6$ m s^{-1}									
Coefficient of restitution, $e = 1$									
0.800	0.0	−0.6	194.5	0.0	0.777	0.0	−0.46	193.5	0.01
0.800	0.0*	0.0	194.5	0.0	0.801	0.278	0.0	150.2	5.30
0.800	0.0*	0.0	194.5	0.0	0.787	0.282	0.0	148.7	6.94
Coefficient of restitution, $e = 0.5$									
0.600	0.0	−0.30	145.9	22.1	0.561	0.0	−0.16	138.4	23.6
0.732	0.263*	0.0	127.6	17.9	0.616	0.166	0.0	129.6	21.2
0.732	0.263*	0.0	127.6	17.9	0.616	0.162	0.0	129.6	21.3
0.732	0.263*	0.0	127.6	17.9	0.613	0.159	0.0	129.7	21.2
Coefficient of restitution, $e = 0.05$									
0.433	0.0	−0.03	102.1	29.3	0.411	0.0	0.0	100.5	29.3
0.569	0.043*	0.0	100.3	29.3	0.406	−0.014	0.0	100.5	29.0
0.569	0.043*	0.0	100.3	29.3	0.405	−0.017	0.0	100.6	29.0
Initial tangential velocity, $v_t = -1.0$ m s^{-1}									
Coefficient of restitution, $e = 1.0$									
0.800	0.0	−2.20	194.5	0.0	0.777	0.0	−2.07	193.5	0.0
1.633	0.846*	0.0	60.7	68.6	1.617	0.855	0.0	65.3	68.2
1.633	0.846*	0.0	60.7	68.6	1.785	0.813	0.0	79.3	50.0
Coefficient of restitution, $e = 0.5$									
0.600	0.0	−1.9	145.9	15.0	0.561	0.0	−1.76	138.4	16.0
1.395	0.905*	0.0	30.4	92.1	1.301	0.911	0.0	31.1	92.5
1.395	0.905*	0.0	30.4	92.1	1.307	0.907	0.0	30.5	92.4
Coefficient of restitution, $e = 0.05$									
0.420	0.0	−0.43	102.1	38.4	0.411	0.0	−1.60	100.46	19.9
1.031	0.987*	0.0	0.0	99.6	1.031	0.988	0.0	3.06	99.9
1.031	0.987*	0.0	0.0	99.6	1.031	0.989	0.0	3.06	99.9

* Impulse ratio, $\mu = \mu_0$.
Oriented at 45°; initial normal velocity $v_n = -1$ m s^{-1}.

problem was not solved here, it is easy to see from the Eqn (4) that if v_{Cn} is zero, then V_{Cn} also must be zero and no impact takes place. Consequently, the claimed inconsistency does not exist for this problem.

Though comparisons are made between results from classical theory and simulation solutions, they are not exhaustive. The question of the accuracy of the classical theory needs additional investigation.

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