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FORMULATION OF RIGID BODY IMPACT PROBLEMS USING GENERALIZED COEFFICIENTS

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Abstract—The equations of motion of a rigid body expressed in terms of impulse and momentum are linear. When applied to rigid body collisions, it is known that the equations of motion are insufficient to provide a solution of the classical impact problem; an additional equation is needed for each unknown impulse component. Using a set of coefficients, a problem formulation is presented that extends Newton's approach for collinear impacts of particles to three-dimensional impact problems. Being linear and algebraic these equations can be solved, providing a set of solution equations in terms of the physical system parameters, initial conditions and the coefficients. A unique feature of these equations is that they are independent of the contact process(es) and apply to all collisions meeting the rigid body assumptions, whether energy is or is not conserved (contact processes may involve the release of stored energy). Certain solution behavior, including the energy change can be found by treating the coefficients as parameters. By imposing work–energy and/or kinematic constraints, coefficients can be bounded to insure realistic solutions. Coefficients are defined for couple-impulses so the approach is not limited to point contact.

Examples are given of the collision of a sphere against a massive barrier (surface). In one, the sphere has an initial cross spin (about its roll–spin axis) and the tangential process is Coulomb friction. Another, including experimental data, is for microspheres (≈ 1–$100\ \mu$m diameter), where the dynamic contact processes are not fully understood. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Methods used for modeling mechanical impacts range widely in complexity and objectives. Stresses in the contact region are often required for design purposes. In other cases velocity changes and energy losses are sought. The latter is of interest here. Even with such restricted objectives different approaches are possible. The simplest by far is with the use of rigid body impact theory because it can result in a purely algebraic approach. The justification for extending an algebraic theory to three dimensions is the model's remarkable generality: the impact dynamics problem can be formulated and solved without specifying the contact process. Newton's approach to the central impact of particles (for moderate to low speed applications) uses the concept of a coefficient of restitution, impulses and linear and angular momentum to calculate velocity changes and determine kinetic energy loss. Not all problem formulations are algebraic. For Coulomb friction, Keller [1] has developed an approach that uses integration of the contact impulse variables. Stronge [2] has shown how Keller's approach can be used over a flat surface using an energetic coefficient of restitution. Maw *et al.* [3] developed a model that treats tangential restitution. Approaches such as these have been developed individually for specific types of interface mechanics, such as dry friction, tangential restitution, and so on. A comprehensive review of the above and other approaches to impact modeling has been prepared by Mac Sithigh [4]. Depending on the contact process and the approach, solution of the model equations can involve mathematical methods ranging from linear algebraic to nonlinear differential equations. An objective here is to present a broad formulation of the impact problem useful for a variety of engineering applications. The approach is not limited to planar or even small contact interfaces. The contact interface between the two colliding bodies may generate forces which can be simulated by a kinematic constraint such as a rapid clamping action or magnetic attachment. Although not pursued directly in this paper, cases can occur where an interface mechanism can release energy such as a spring-loaded device. The formulation using generalized coefficients is intended to apply to all types of contact processes.

This is not a first introduction of this approach but rather a more detailed discussion that includes some subtle attributes, particularly the stationary property of energy loss. In fact, the method has already been extended to collisions of pseudo-rigid bodies by Cohen and Mac Sithigh [5]. Some examples are also included to illustrate the method, one in particular contains experimental data and shows how the solution equations can serve as a generic impact model for examining experimental data from collisions.

In the context of impact it is known that Newton's second law (in the form of impulse and momentum and using only initial and final values of all variables) provides an insufficient number of equations for a solution of a problem. The approach here is to derive additional equations, each based on the definition of a coefficient, sufficient in number to supplement Newton's laws and provide equations equal in number to the unknowns. The number of coefficients is equal to the number of (unknown) impulse components at the interface. In some applications, see Brach [6], external impulses may need to be evaluated in an iterative fashion using kinematic constraints. The complete set of equations, called the system equations, is linear in the unknowns and is independent of the contact interface mechanics. The solutions of the system equations are referred to as the solution equations. Values, ranges of values, or bounds on the coefficients and corresponding solution can then be determined, based on the physical nature of the contact process and work–energy principals. As Newton's kinematic coefficient for central impacts (and Stronge's energetic coefficient) are not material constants, the coefficients defined here also must be evaluated experimentally or analytically related to the contact process(es). Specific solutions for given processes and initial conditions in some cases may require integration or piecewise evaluation of variables to determine certain coefficients defined in the formulation of the problem.

Some features of the formulation require discussion. The first concerns reaching a full solution of the rigid body impact problem by evaluating the solution equations for given initial conditions and specific coefficient values. As a general formulation (independent of particular contact processes), the coefficients are independent constants whose values determine a solution of the system equations. For some applications the coefficients may be determined directly such as from known constraints, experimental data or from an independent analytical procedure such as a dynamic finite element analysis. For example, Jäger [7] determines the coefficient of tangential restitution for Hertzian elastic contact. In cases such as these, substitution of the coefficients into the system equations provides a solution. For specific contact processes, the coefficients can lose their independence and their evaluation may be complicated. Whatever way coefficients are found, their use in the system equations provides a solution to the impact problem that satisfies Newton's laws.

It is often thought that only small deformations contained or restricted to infinitesimal contact regions can be treated by rigid body impact theory. An inference follows that since the contact region is infinitesimally small, significant contact moments (couples) cannot develop and thus should not be included in rigid body impact theory. In fact, rigid body theory can apply to problems with other than point contact. Figure 1 shows examples of a soft sphere and a hard annulus, each colliding against a relatively rigid flat surface. When the sphere has initial spin, ω_n, a moment coefficient associated with M_n can be defined and used (the moment coefficient is likely to be related to Newton's coefficient e_n and illustrates potential dependence discussed earlier). Both coefficients can depend on the initial conditions.† The separation of rigid body impact dynamics and the process of coefficient evaluation is always possible as long as the assumption of short duration contact is met.

Examples of where the algebraic simplicity of this approach can provide a distinct advantage includes:

†Some may say that coefficients that depend on initial conditions (and are not material constants) have little utility. This view does not recognize that when applied to nonlinear contact processes (as most impact problems are), the dependence on initial conditions is actually necessary in order to provide a useful model.

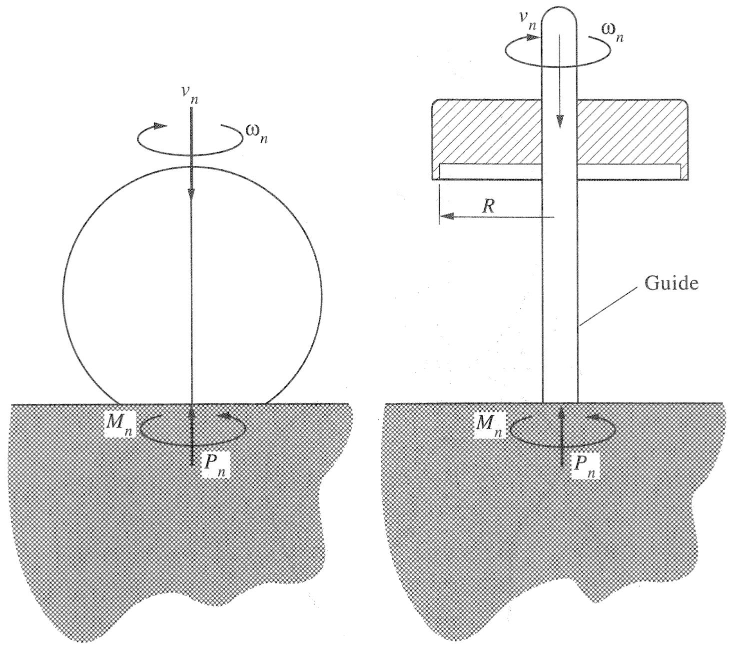

Fig. 1. Sketches representing a soft sphere and a hard annulus colliding with another object, collisions that can be modeled using rigid body impact theory.

1. fitting of experimental collision data, particularly when the contact processes are obscure or unknown;
2. design of a mechanical system whose overall motion includes collisions including optimal design problems where the coefficient values can be optimized in parametric space;
3. Monte Carlo simulation of impact processes;
4. analysis of vibratory impact processes including unstable and chaotic motions;
5. collisions where the process is inherently complex (such as collisions of vehicles);
6. when an impact is a part of a broader physical problem and model simplicity is necessary.

However, it must be realized that the rigid body approach to impact itself has limitations. These can be significant. As seen by Stoianovici and Hurmuzlu [8] for a long slender rod whose end strikes a hard surface, changing the angle of incidence of the rod excites modes of vibration differently and results in a significant change in the coefficient of restitution, owing simply to changes in the system configuration.

2. SYSTEM EQUATIONS

The approach in this paper consists of deriving a set of equations from basic principles of mechanics augmented by equations that define coefficients. The resulting equations, called *system equations*, are linear in the unknown velocity and impulse components. A set of *solution equations* is found. It must be recognized that the solution equations do not directly provide solutions for specific contact processes (such as Coulomb friction or viscous friction) until the coefficients are related to these processes. However, with only defining some, or none, of the contact processes work–energy concepts and/or kinematic constraints can be applied directly to provide bounds on the values of the coefficients. What this formulation does is postpone the process of relating the coefficients to the contact processes. In some applications this presents no advantage; in general, it does. Experimentally determined coefficients, of course, are an example of the latter. This approach is now covered for the three-dimensional impact of two

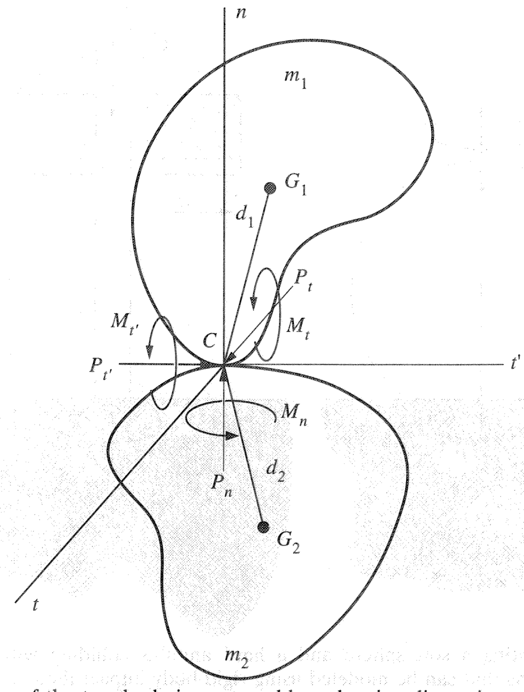

Fig. 2. Diagram of the two body impact problem showing dimensions and coordinates.

rigid bodies with masses m_1 and m_2 as sketched in Fig. 2. A single tangential plane containing a common contact point C is chosen. A normal to this plane is chosen through the point C. In some applications the choice of C may not be unique and can may require estimation or iteration. If the mass centers of the two bodies lie on this axis throughout the collision, it is called a central collision. By choosing two perpendicular directions within the tangent plane, a triad with coordinates n, t and t' is formed.

2.1 Equations of impulse and momentum.

For a contact duration over a time interval τ_1 to τ_2, where τ is time, the change in linear momentum of masses m_i, $i=1,2$, is

$$m_i\mathbf{V}_i - m_i v_i = (-1)^{i-1}\mathbf{P}, i=1,2 \tag{1}$$

Capital symbols represent variables at the end of contact, τ_2, and variables in lower case symbols represent values at the initiation of contact, τ_1. Vector quantities are indicated by bold characters. The velocities in equation (1) are of the mass center, $\mathbf{V_1}$ with components (V_{1n}, V_{1t}, $V_{1t'}$) and $\mathbf{V_2}$ with components (V_{2n}, V_{2t}, $V_{2t'}$). The total impulse is $\mathbf{P}$ has components (P_n, P_t, $P_{t'}$). The relationships between angular momentum and the moments of the impulses is

$$\mathbf{H}_i - \mathbf{h}_i = (-1)^{i-1}\mathbf{M} + \mathbf{d}_i \times (-1)^{i-1}\mathbf{P}, i=1,2 \tag{2}$$

where the angular momentum components of $\mathbf{H_i}$ and $\mathbf{h_i}$ involve the bodies' inertia tensors. The vector $\mathbf{M}$ represents the impulses of any moments (couples) acting over the contact region and $\mathbf{d_i}$ is the vector from each mass center to point C. Equations (1) and (2) form four linear vector equations, 12 scalar equations, containing a total of 18 unknowns. Six more equations are needed to complete the system equations, one each for the components of $\mathbf{P}$ and $\mathbf{M}$.

2.2 Coefficients

The most well-known coefficient used in impact analysis is the kinematic coefficient of restitution (usually indicated by the symbol e) for central collisions and attributed to Newton. It is the negative ratio of the rebound to approach speeds normal to a tangent plane. It is a kinematic quantity but with a well-known relationship to energy loss; energy principles require that $0 \le e \le 1$. The use of coefficients in the three-dimensional problem is not quite as simple and has been discussed by Stronge [9] and Brach [10]. In a more general fashion, it is possible to define a triplet of kinematic coefficients of restitution e_n, e_t and $e_{t'}$ (corresponding to the three coordinate directions) in a vector $\Delta \mathbf{V}_{Cr}$ with a jth component $V_{Crj} + e_j v_{Crj}$, $j = n,t,t'$ such that

$$\Delta \mathbf{V}_{Cr} = 0 \tag{3}$$

where $\mathbf{V}_{Cr}$ is the final relative contact velocity and $\mathbf{v}_{Cr}$ is the initial relative contact velocity. It is important to recognize that the coefficients in equation (3) are simply kinematic constants governing the ratio of final-to-initial velocity components. Their relationship to energy loss is discussed later. Three system equations remain to be developed, related to the moment impulse components. A vector $\mathbf{Q}$ is defined with a jth component, $Q_j = e_{mj} M_j + (1 + e_{mj}) \bar{I}_j (\Omega_{2j} - \Omega_{1j})$ such that

$$\mathbf{Q} = 0 \tag{4}$$

where the coefficients e_{mj}, $j = n,t,t'$, are kinematic constants related to relative rotational motion and the Ω_{ij}s are the final angular velocity components of body i about axis j. The scalar quantity $\bar{I}_j$ is an inertial constant equal to the product divided by the sum of the principal moments of inertia about the j axis of each body, respectively. Examination of each component shows that if an $e_{mj} = 0$, the final relative angular velocity about the j axis must be zero. Similarly, if $e_{mj} = -1$, then the corresponding moment impulse M_j must be zero.† Frequently an impact problem is solved for conditions of point contact where moments or couples cannot develop. This corresponds to the conditions that $e_{mj} = -1$, $j = n,t$ and t'. In any case the three moment coefficients are constants carried through to the solution equations. When the problem does not directly involve rotational kinematic constraints (that is, the e_{mj}s are not known directly), experimental information or the physical process of couple generation and the energy loss in the collision must be used to determine appropriate values of the coefficients for a solution. This topic admittedly needs additional study with only a simple example is presented here corresponding to rolling friction or drag.

The above equations (1)–(4), are linear and contain 18 unknowns, consisting of 12 final velocity components and 6 force and moment impulse components. Analogous to the classical Newtonian coefficient, e, each of the coefficients e_j and e_{mj} defined above has the ability to provide a measure of "restitution" associated with velocity components along the jth coordinate. For example, tangential restitution in planar collisions has been observed and measured by Maw, Fawcett and Barber, [3], its effect modeled by Brach [12] and applied to simulation of granular flow by Moreau [13]. If the scope of the problem is reduced to the point contact case by considering only when the couples M_j are absent, the number of unknowns reduces to 15 and equation (4) is no longer needed.

For some problems it is convenient to introduce kinetic coefficients, μ_t and $\mu_{t'}$, in place of e_t and $e_{t'}$. An equivalence between the coefficients exists (discussed later) and convenient combinations of the types of coefficients can be used. equation (3) is replaced to introduce the such a combination of coefficients. The first component of equation (3) is retained, that is

$$V_{Crn} = -e_n v_{Crn} \tag{5}$$

The ratio of the final tangential impulse components, P_t and $P_{t'}$ to the final normal impulse component provides kinetic coefficients μ_t and $\mu_{t'}$, such that

$$P_t = \mu_t P_n \tag{6}$$

$$P_{t'} = \mu_{t'} P_n \tag{7}$$

Without specifying a priori any specific tangential physical processes, μ_t and $\mu_{t'}$ are constants (kinetic coefficients) corresponding to the values of the ratios of the final impulse components. Equations (5)–(7) are alternatives to and replace equation (3). As a result of the introduction of

†Some of these definitions are arbitrary and have been chosen to facilitate application of rotational kinematic constraints on the solution equations. Other definitions are possible and may possess other desirable features.

the alternative coefficients $(e_n, \mu_t, \mu_{t'})$, two sets of system equations and solution equations now exist. These two solution sets for the same set of initial velocities and interface properties must obviously bear a relationship. In fact, equating the solutions for each of the tangential velocity components V_j, $j = t,t'$, provides equivalency relationships between the coefficients e_t, μ_t, $e_{t'}$ and $\mu_{t'}$. This equivalence has been explored for collinear impacts by Brach [12]. There may be instances where kinetic coefficients associated with the moment impulses, M_n, M_t and $M_{t'}$, may be more convenient than the kinematic coefficients in equation (4). An example is in Brach [12].

To summarize, a set of 18 linear system equations has been derived for the full three-dimensional or 15 variables in the point contact impact problem $(M_n = M_t = M_{t'} = 0)$. Solution of such a set of equations (but with specific coefficients) has been outlined by Cohen and Mac Sithigh [4] and others. Solution equations provide a solution for a given set of initial velocities once the coefficients are known. The solution provides all final velocity components and all impulse components. This means that $V_{ij}(v_{ip}, \omega_{ij}, e_q, \mu_r)$, $\Omega_{ij}(v_{ij}, \omega_{ij}, e_q, \mu_r)$ and $P_j(v_{ij}, \omega_{ij}, e_q, \mu_r)$ are known, $i = 1,2$ and $j = n,t,t'$.

3. KINETIC ENERGY LOSS

One of the advantages of the above approach is that once the system equations are solved, any aspect of system behavior including the kinetic energy loss can be expressed in terms of the initial conditions and the coefficients. It also permits work–energy constraints to be applied. This permits an exploration of how the coefficients control energy loss, that is, how impulse components do work and dissipate energy. This exploration is convenient and sometimes necessary to provide values or bounds on the coefficients. In some problems, the work of individual impulse components is important and must be examined separately. An important example is when determining the work of the normal impulse during approach and rebound for use with Stronge's [9] normal energetic coefficient. At this point the energy loss expression for the point contact problem is examined to illustrate a useful feature of the impulse ratios.

When couple impulses are absent, the final velocity components at point C can be expressed in terms of the final mass center velocities and the final angular velocities. equation (2) can then be used to eliminate the angular velocities leaving the contact velocity components expressed in terms of the impulse components. This provides the following matrix equation

$$\mathbf{V}_{Cr} - \mathbf{v}_{Cr} = [\mathrm{a}]\mathbf{P} \tag{8}$$

where the matrix [a] depends upon inertial properties and the distances from the mass centers to point C. Using equation (8) the vector impulse can be written in the form

$$\mathbf{P} = [\mathrm{a}]^{-1}(\mathbf{V}_{Cr} - \mathbf{v}_{Cr}) \tag{9}$$

Using equation (5), the first component of the contact velocity change can be written as $V_{Crn} - v_{Crn} = -(1+e)v_{Crn}$. A limiting case is now defined where simultaneously, there is no normal rebound $(e_n = 0)$ and relative tangential motion ceases at or prior to the end of contact; this corresponds to $(e_n, \mu_t, \mu_{t'}) = (0, \mu_{t0}, \mu_{t'0})$. These limiting or critical impulse ratios can always be found from the solution equations and by setting the final relative tangential velocities to zero, solving for the corresponding impulse components, P_{n0}, P_{t0} and $P_{t'0}$ and computing their ratios. Using these, the kinetic energy loss, T_L, of the impact can be written as

$$T_L = -\tfrac{1}{2}P_n(P_n\{1\,\mu_t\mu_{t'}\}[\mathrm{a}]\{1\,\mu_t\mu_{t'}\}^T - 2P_{n0}\{1\,\mu_t\mu_{t'}\}[\mathrm{a}]\{1\,\mu_{t0}\mu_{t'0}\}^T) \tag{10}$$

The superscript T indicates the transpose of the vectors in braces. Recall that from the solution equations P_n is a function of initial velocities and the coefficients. For some special cases (including point mass, central rigid body impacts and some constrained problems) P_n is independent of the impulse ratios. In those cases, the quadratic form in the impulse ratio coefficients of the first term of equation (10) and the linear form of the second term shows that the kinetic energy loss is stationary for $(e_n, \mu_t, \mu_{t'}) = (e_n, \mu_{t0}, \mu_{t0'})$; the peak energy loss occurs for $(e_n, \mu_t, \mu_{t'}) = (0, \mu_{t0}, \mu_{t0'})$. This was noted by Brach [12]. For certain applications such as dry friction the kinetic coefficients are not independent; rather $\mu_t = \mu \sin\eta$ and $\mu_{t'} = \cos\eta$. The coefficient μ is the resultant impulse ratio, η is the angle made in the tangent plane by the resultant final tangential impulse and $\mu^2 = \mu_t^2 + \mu_{t'}^2$.

The same concepts just discussed apply to the full three-dimensional problem including the form of the expression for the kinetic energy loss. Finding an equation corresponding to (10) requires two additional steps. First, energy loss in the presence of the moment, or couple, impulses must be written. Then the moment impulses M_n, M_t and $M_{t'}$ must be expressed in terms of the final velocities by using equations (2) and (4). These are not presented.

For dry friction, Stronge [9] has shown that the kinematic and kinetic coefficients introduced and used above are not independent. In place of e_n, Stronge introduces a energetic coefficient, E^2, which is defined as the negative ratio of the work done by P_n during rebound to the work done by P_n during approach. The energetic coefficient and the friction coefficient, f, form an independent pair of parameters for impact. Consequently, for dry friction $e_j = e_j(E,f)$, $e_{mj} = e_{mj}(E,f)$ and $\mu_j = \mu_j(E,f)$. This is inherent in the approach presented here and is a result of the trade off made at the beginning whereby treating process nonlinearities are deferred until after the solution equations and kinetic energy loss expressions are available. These functional relationships between coefficients can sometimes be found analytically, a topic dealt with in the examples. It is seen that since impact coefficients such as Stronge's energetic coefficient depend on the initial conditions (i.e. it is not a material constant) and the impulse ratios (and tangential coefficients) have limiting values that depend on the initial conditions, that the generalized coefficients depend on initial conditions. This is discussed further in the following.

Finally, it should be noted that for the solution of some problems it is convenient and sometimes necessary to define partial impulse ratios, such as by Brach [12]. These are ratios of the tangential impulse components to the normal impulse component over specific portions of the contact duration such as approach, rebound and other events such as when sliding stops or reverses.

4. EXAMPLES

Two examples are summarized to illustrate various features of the approach discussed above. The first is a point contact problem of a sphere hitting a flat rigid barrier with initial spin about the projection in the tangential plane of its initial velocity vector. A solution of this problem using a different approach is presented by Stronge [2]. The second is a full three-dimensional problem reduced to planar conditions where a sphere collides with a massive barrier with a couple permitted at the contact area. Experimental data for this type of problem is introduced to illustrate the application.

4.1 *Point contact of a sphere against a flat rigid barrier*

In this problem, illustrated with the free body diagram in Fig. 3, the moment impulses M_n, M_t and $M_{t'}$ at C are all zero. The system equations derived above apply by dropping all equations with subscript 2, which reduces the number from 18 to 12. With zero moment impulses, $e_{mj} = -1$, $j = n,t,t'$, equation (4) becomes trivial and the number is reduced to 9. With $M_n = 0$, no change in ω_n can occur and $\Omega_n = \omega_n$, leaving 8 unknowns and 8 system equations. The solution equations give the final velocities

$$V_n = -e_n v_n \tag{11}$$

$$V_t = v_t - (\mu_t/\mu_{t0})\lambda_{t'} v_{Ct}, \; |\mu_t| \le |\mu_{t0}| \tag{12}$$

$$V_{t'} = v_{t'} - (\mu_{t'}/\mu_{t'0})\lambda_t v_{Ct'}, \; |\mu_{t'}| \le |\mu_{t'0}| \tag{13}$$

$$\Omega_t = \omega_t - (\mu_{t'}/\mu_{t'0})\gamma_t v_{Ct'}, \; |\mu_{t'}| \le |\mu_{t'0}| \tag{14}$$

$$\Omega_{t'} = \omega_{t'} - (\mu_t/\mu_{t0})\gamma_{t'} v_{Ct}, \; |\mu_t| \le |\mu_{t0}| \tag{15}$$

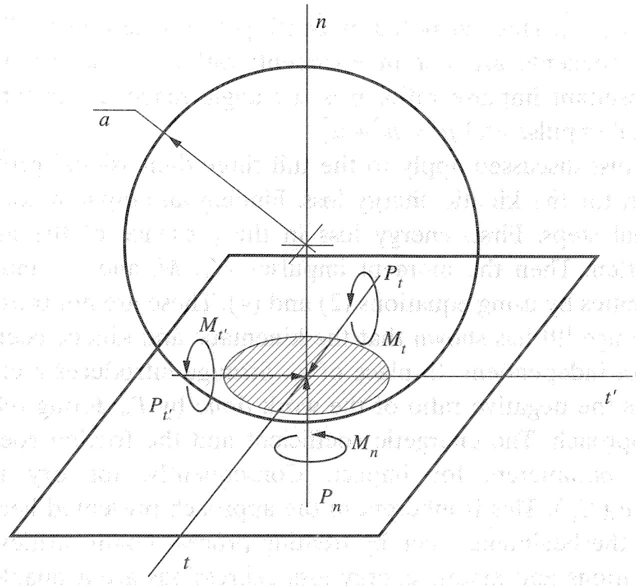

Fig. 3. Dimensions and coordinates of a sphere colliding with a flat rigid surface.

where for a sphere $\lambda_j = 2/7$, $\gamma_j = 5/7a$ and $\mu_{j0} = 2v_{Cj}/7v_n(1+e_n)$, $j = t, t'$. The energy loss expression is

$$T_{\mathrm{L}} = \tfrac{1}{2}m(1-e_n^2)v_n^2 + \tfrac{1}{2}m\lambda_{t'}(\mu_t/\mu_{t0})(2-\mu_t/\mu_{t0})v_{Ct}^2 + \tfrac{1}{2}m\lambda_t(\mu_{t'}/\mu_{t'}0)(2-\mu_{t'}/\mu_{t'0})v_{Ct'}^2,$$
$$|\mu_t| \le |\mu_{t0}| \text{ and } |\mu_{t'}| \le |\mu_{t'0}| \tag{16}$$

Corresponding to equation (10), P_n has been eliminated and it can be seen that the energy loss is a maximum at $(e_n, \mu_t, \mu_{t'}) = (e_n, \mu_{t0}, \mu_{t'0})$. If the contact surface is flat and tangentially isotropic and the process is dry friction, Stronge [2] shows that the direction of slip is a constant. Then $\mu_t = \mu\cos\eta$ and $\mu_{t'} = \mu\sin\eta$ and $\mu^2 = \mu_t^2 + \mu_{t'}^2$. The angle η is the arctangent of $\mu_{t'}/\mu_t$. The energy loss is

$$T_{\mathrm{L}} = \tfrac{1}{2}m(1-e_n^2)v_n^2 + \tfrac{1}{2}m(2/7)(\mu/\mu_0)(2-\mu/\mu_0)(v_{Ct}^2 + v_{Ct'}^2),\ |\mu| \le |\mu_0| \tag{17}$$

where $\mu_0 = 2(v_{Ct}^2 + v_{Ct'}^2)^{1/2}/7v_n(1+e_n)$. This now is a central impact so $0 \le e_n \le 1$. For a solution for a specific value of friction coefficient, f, the impulse ratio μ is set equal to f (with the sign of μ_0) as long as $f \le |\mu_0|$, otherwise $\mu = \mu_0$. Note that when $e_n = 0$ and $\mu = \mu_0$, the sphere does not rebound and its final motion is that of rolling on the surface. Without the moment impulses it is not possible for the sphere to lose all of its kinetic energy even though relative velocity at the contact point ceases.

4.2 Planar impact of a sphere with a barrier with a adhering surface

In this problem the surface has an adhesive capability and can fully seize the sphere. To make this possible, it is assumed that adhesion acts over a finite contact area and can retard rolling of the sphere† as well as its rebound. The coefficient e_n represents the combined effects of dissipation due to deformation as well as adhesion normal to the surface. The free body diagram in Fig. 3 applies. For this example initial conditions are such that no motion or forces are generated along the t' coordinate and no moment about the normal axis exists. That is, $\omega_n = \omega_t = v_{Ct'} = v_{t'} = 0$ and $\omega_{t'} = \omega$. The solution equations are

$$V_n = -e_n v_n \tag{18}$$

$$V_t = v_t - (\mu/\mu_0)\lambda(v_t + e_{mt'}a\omega) \tag{19}$$

$$\Omega_{t'} = -e_m\omega - e_m(\mu/\mu_0)a\lambda(v_t + e_m a\omega)/k^2 \tag{20}$$

†Rolling resistance is typically ignored during impact but its introduction is not entirely new. See Sinitsyn [11] for a different approach.

$$P_n = -m(1+e_n)v_n \tag{21}$$

$$P_t = \mu P_n \tag{22}$$

$$M_{t'} = -(1+e_{mt'})(mk^2\omega - aP_t) \tag{23}$$

where $k^2 = 2a^2/5$, $\mu_0 = \lambda(v_t + e_{mt'}a\omega)$ and $\lambda = (1 - e_{mt'}a^2/k^2)^{-1}$. The energy loss is

$$\begin{aligned} T_L(e_n,\mu,e_{mt'}) = {} & \frac{1}{2}m(1-e_n^2)v_n^2 + \frac{1}{2}mk_2(1-e_{mt'}^2) + \frac{1}{2}m\lambda(\mu/\mu_0)(v_t + e_n a\omega)[2(v_t - e_n^2 a\omega) \\ & - (\mu/\mu_0)\lambda(v_t + e_{mt'}a\omega)(1 + a^2 e_{mt'}2/k^2)] \end{aligned} \tag{24}$$

An interesting and common special case of energy loss is when a moment impulse cannot develop, that is, when $e_{mt'} = -1$. This has an energy loss expression

$$T_L(e_n,\mu,-1) = \frac{1}{2}m(1-e_n^2)v_n^2 + \frac{1}{2}m\lambda(\mu/\mu_0)(v_t - a\omega)[2-(\mu/\mu_0)] \tag{25}$$

which is the planar version of equation (16), in the previous example. Another special case, one of particular interest is when the sphere completely sticks to the surface. In this case

$$T_L(0,\mu_0,0) = \frac{1}{2}mv_n^2 + \frac{1}{2}mv_t^2 + \frac{1}{2}mk^2\omega^2 \tag{26}$$

found from the condition $(e_n, \mu, e_{mt'}) = (0, \mu_0, 0)$ and represents an impact where the sphere fully adheres to the surface and loses all of its initial kinetic energy. This is an example of the children's game where a soft, fabric ball is thrown against a target covered with Velcro. The ball fully attaches to the target at the location of impact. The inclusion of a moment impulse and its corresponding coefficient is necessary to model this process; point contact theory alone is insufficient.

Another aspect of the problem is to examine the case where the final conditions are such that the sphere does not rebound ($V_n = 0$), is rolling ($\mu = \mu_0$) but still has a small amount of (rotational) kinetic energy at time τ_2. The energy loss in this case is found from equation (24) with $(e_n, \mu, e_{mt'}) = (0, \mu_0, e_{mt'})$. Consider the energy loss normalized to the initial kinetic energy, T, which is 1 minus the fraction of remaining energy. That is, $T_L^* = 1 - T_R^*$. For these conditions

$$T_R^* = \frac{e_{mt'}^2}{(k^2/a^2 - e_{mt'})^2}\,\frac{(v_t + k^2\omega/a)(1 + k^2/a^2)}{(v_n^2 + v_t^2 + k^2\omega^2)} \tag{27}$$

The quantity T must be bounded below by zero and above by the fraction of the remaining kinetic energy under the condition of no moment impulse. This can be obtained from equation (24) as $T_L(0, \mu_0, -1)$. Consequently

$$0 \le T_R^* \le \frac{(v_t + k^2 a\omega/a^2)(1 + k^2/a^2)}{(1 + k^2/a^2)^2(v_n^2 + v_t^2 + k^2\omega^2)} \tag{28}$$

This serves as a bound on the moment coefficient, $e_{mt'}$. Figure 4 illustrates the functional dependence of T on $e_{mt'}$ from equation (27). Since $e_{mt'} = -1$ (with $e_n = 0$ and $\mu = \mu_0$) represents the final condition of unrestrained rolling and $e_{mt'} = 0$ represents full sticking, the condition $-1 \le e_{mt'} \le 0$ represents rolling after a rotationally dissipative impact. Positive values of $e_{mt'}$ represent a rotational elastic contact phenomenon, something not associated solely with "sticking" and not applicable to this example.

As a final part of this example, experimental data is presented to illustrate the impulse ratio coefficient, its limiting values and how is used to represent a tangential process. The dynamic contact mechanics associated with collisions of very small particles, in the range 1–100 μm as studied by Dunn *et al.* [14] is not fully understood. Force systems that act normal to the contact surfaces such as van der Waals adhesion, capillary and electrostatic forces become significant relative to elastic body forces and weight. These adhesive forces can inhibit rebound and even

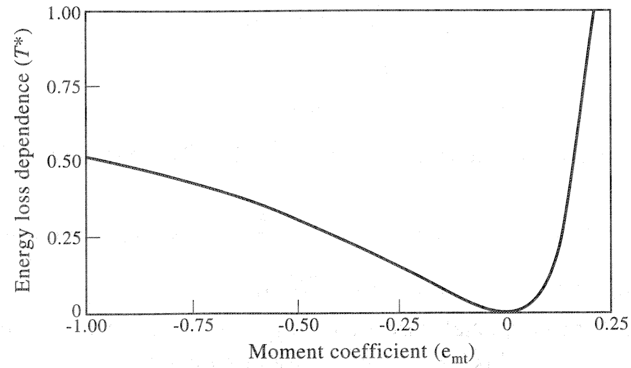

Fig. 4. Functional dependence of the remaining energy, T_R^*, on the moment coefficient (equation (27)) of a rolling sphere with no rebound.

cause attachment to occur. In some cases this is desirable (such as filtration and the use of toner powders in xerography) in others it is not (contamination of microchips during manufacturing). Not only are dynamic normal forces not well understood but it is not even known if the Coulomb friction model is appropriate. It is seen that the impulse ratio found in the above equations has a critical value, dependent on the initial conditions. For this application—if Coulomb friction is applicable—the impulse ratio behaves as illustrated by the solid curve segments in Fig. 5. The constant dynamic friction coefficient is f and $\mu = f$ as long as the angle of incidence is low enough so that sliding continues throughout the duration of contact. Figure 5 also shows corresponding data averaged over numerous oblique impacts of microspheres against a molecularly smooth flat surface for each of several angles of incidence. The total initial velocity was held constant as the angle of incidence was varied; 90° is normal incidence. The coefficient of normal restitution and the value of the coefficient of friction for the solid curves are chosen to agree with the data and are $e_n = 0.79$ and $f = 0.14$. The experimental data show reasonably good agreement with the Coulomb friction model except at the lowest experimental approach angle where the impulse ratio value is low. (Analysis of the data shows that the difference of the experimental point from $f = 0.14$ is statistically significant.) This point also corresponds to the lowest initial normal velocity, which is where the adhesion forces have the greatest significance. All of this implies that interaction between the tangential and adhesion forces may exist and indicates that Coulomb's friction model applies in the presence of adhesion but some other effect is present at very low normal velocities.

Three data points appear near but above and below the $\mu = \mu_0$ curve. Their deviation from the theoretical curve represents a bias owing to the presence of a significant initial angular velocity. The experiments that produced this data were unable to shed any light on the significance of rolling dissipation and the moment coefficient. The reason is that current laser

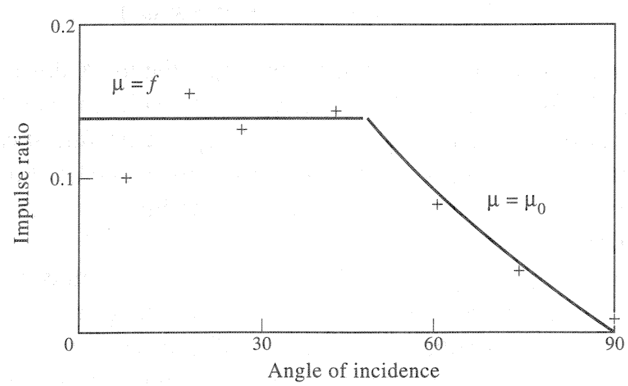

Fig. 5. Behavior of the impulse ratio (solid curves), μ, for the Coulomb friction model with zero initial angular velocity; data is from microsphere–surface impact experiments in the presence of adhesion forces, from Dunn et al. [14].

techniques do not allow direct dynamic measurement of angular velocities or their changes for particles in the size range being studied.

5. SUMMARY AND CONCLUSIONS

A formulation of the three-dimensional impact problem for collisions of rigid bodies has been outlined. It develops a set of system equations using a combination of Newton's laws and definitions of combinations of kinetic and kinematic coefficients. Since these equations are always algebraic and linear, a set of solution equations is always possible and provides expressions for the final velocities, impulse components and kinetic energy loss in terms of the initial conditions and coefficients. These equations are completely general for all possible physical processes at the contact interface such as friction, restitution and indentation, etc.

Advantages of this approach over past formulations of the impact problem are the generality, the availability of solution equations and the independence of the contact process. For cases where the contact process is governed by kinematic and kinetic conditions, or both, the solution equations directly provide solutions to the problem. This occurs in such applications as machine design including analysis, synthesis and optimization of mechanisms. For example, the solution equations directly provide the energy loss for a mechanism designed to attach at the contact point. The equations can provide the energy gain necessary to achieve given dynamic objectives. Other cases where direct solutions are possible include when impulse components and relative contact velocities have been determined experimentally or through other methods of analysis such as finite element procedures.

Whenever the rigid body approach to impact is appropriate the solution equations can be used to examine bounds on the motion by examining limiting or critical values of the coefficients. In this way classes of solutions can be found and examined. When the contact interface process (such as dry friction) dictates a nonalgebraic solution technique, the above method adds no additional difficulty or computational complexity. Computationally, the approach lends itself nicely to a general computer simulation. The system equations can be solved as part of a main routine with specific interface processes relegated to different subroutines.

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